On Convergence of Random Series

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Outline

Numerical Series

Series

- Sequence $a_1, a_2, \ldots$ real numbers.
- **Series**, the "infinite sum" denoted by "$\sum_{m=1}^{\infty} a_m$", with terms $a_1, a_2, \ldots$.

Convergence

Infinite sum $=$ Limits of finite sums.

- Sequence of partial sums $S_n = \sum_{m=1}^{n} a_m$.
- Series **converges** $\Leftrightarrow$ $\lim_{n\to\infty} S_n$ exists as a real number, the **sum** of the sequence.
- Diverges otherwise.

Examples

- Geometric, $a_n = q^{n-1}$.
- Harmonic, $a_n = \frac{1}{n}$.
- Alternating harmonic, $a_n = \frac{(-1)^{n+1}}{n}$.
- Other? Taylor series, and more fancy stuff.

No closed form in general. Sometimes can get estimates, or, more generally can determine if converges or not.

Convergence tests

Tools to determine convergence. Many... Here are a few.

Theorem 1 (Comparison)

Suppose $\sum_{n=1}^{\infty} b_n$ *is converges and* $|a_n| < b_n$. *Then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Theorem 2 (Condensation - substitution)

Suppose $a_n \geq a_{n+1} \geq \ldots 0$. *Then* $\sum_{n=1}^{\infty} a_n$ *converges* $\Leftrightarrow$ $\sum_{n=1}^{\infty} 2^n a_{2^n}$ *convegres.*

Examples

- $a_n = q^{n-1}$. Then $S_n = \frac{1-q^n}{1-q}$ and therefore $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow |q| < 1$.
- $a_n = n^{-p}$. Then $2^n a_{2^n} = 2^n 2^{-pn} = (2^{1-p})^n = q^n$. Therefore converges iff $p > 1$.
- Try: $a_n = \frac{1}{n \ln(1+n)}$.

Theorem 3 (Dirichlet's test - summation by parts)

Suppose $b_n \searrow 0$ *and the sequence* $(s_n : n = 1, 2, \ldots)$ *is bounded. Then* $\sum_{n=1}^{\infty} b_n(s_{n+1} - s_n)$ *converges.*

Examples

- Let $a_n = \frac{(-1)^{n+1}}{n}$. Apply with $b_n = \frac{1}{n}$ and $(s_n : n = 1, 2, \ldots) = (0, 1, 0, 1, \ldots)$.
- Try: $a_n = \frac{\sin n}{ln(1+n)}$.

Random Series

What's the deal?

- We **sample** a_n randomly.
- Each **realization** of the sampling yields a (possibly) different series.

Example

- Toss a fair coin repeatedly.
- Set
$$H_n = \begin{cases} 1 & \text{n'th toss is } H \\ 0 & \text{n'th toss is } T \end{cases}$$
- Set $a_n = 2^{-n} H_n$
- The series is
$$\sum_{n=1}^{\infty} a_n = \frac{H_1}{2^1} + \frac{H_2}{2^2} + \frac{H_3}{2^3} + \dots,$$
essentially randomly picking some of the terms of the geometric sequence ($q = \frac{1}{2}$).
- $\Rightarrow$ Converges, due to Theorem 1.
- But the sum can be **anywhere** between 0 ($0 = H_1 = H_2 = \dots$) and 1 ($1 = H_1 = H_2 = \dots$).

The series as a Random Variable

Discussion

- Our random series $\sum_{n=1}^{\infty} \frac{H_n}{2^n}$ always converges.
- Estimating its sum? Nothing beyond the trivial bounds 0 and 1.
- Enter probability.

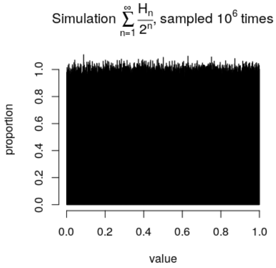
Probabilistic viewpoint

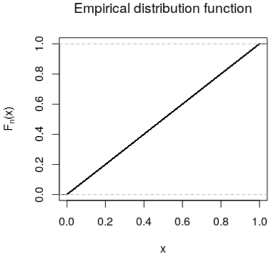
- Switch to a "statistical" perspective.
- Though we don't know what the outcome of the first n tosses will be, we do know all 2^n outcomes have the same probability of appearing.
- So, at least theoretically, we can find the **probability** that the sum lies some interval.
- What's the probability that the sum will be in the interval $[0, 1/2)$? In the interval $[0, 1/4]$? Equal to $\frac{3}{4}$? Between two dyadic numbers?

Bottom line

1. Consider the sum as a function of the "random" realization - an object known as a **random variable**, and
2. Look at the probability this random variable lies an any interval - the **distribution of the RV**.

Simulations





Discussion

- The histogram is a bit noisy, so I added a graph of the corresponding distribution function.
- What is your conclusion?
- Indeed, the distribution of the series $\sum_{n=1}^{\infty} \frac{H_n}{2^n}$ is uniform on $[0, 1]$.
- This gives a bridge between discrete RVs and continuous RVs. Every RV can be generated from an infinite sequence of fair coin tosses.

More simulations

An interesting example

- Let's change from $\sum_{n=1}^{\infty} \frac{H_n}{2^n}$ to $\sum_{n=1}^{\infty} \frac{2H_n}{3^n}$.
- The 2 in numerator to make sure we cover the same range of $[0, 1]$ ($\sum_{n=1}^{\infty} \frac{1}{3} = \frac{1}{2}$).
- What do you think?

More simulations

An interesting example

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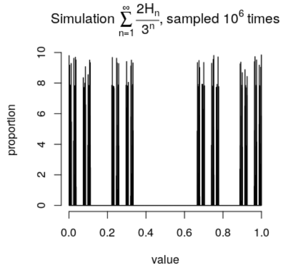
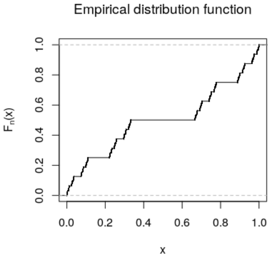




Discussion

- Here the histogram is far from smooth, and again, the picture is much clearer if we look at the empirical CDF.
- The CDF of this RV is the Cantor function.
- This is an example of a RV which is continuous, but has no density.

Other random series?

Recall

- The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- Yet, the alternating signs series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

A random "version"

- Same fair coin, same H_n
- Form the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{H_n}}{n}.$$

- Much tougher than our previous case.
- Diverges/converges for some "freak" realizations, but what happens in the "bulk"?

Some framework

Independent RVs

The RVs $X_1, X_2, \ldots$ are **independent** if information on any of them does not alter the distribution of the other.

Examples

- The RVs $H_1, H_2, \ldots$ from our examples, are independent.
- $f_1(X_1), f_2(X_2), \ldots$ where $X_1, X_2, \ldots$ are independent and $f_1, f_2, \ldots$ are functions.
- Partial sums $X_1 = H_1, X_2 = H_1 + H_2, \ldots$ are not independent.If $X_2 = 2$, then necessarily $X_1 = 1$, although $P(X_1 = 1) = \frac{1}{2}$.

Events

An **event** is a collection of realizations.

1. All but finitely many tosses are H.
2. Any finite pattern appears infinitely many times.
3. The proportion of H in first n tosses converges to the constant c.
4. The random series converges.

Almost sure

- An event holds **almost surely** if its probability is 1 ("the bulk").
- It does not necessarily mean the event contains all realizations!

0-1

Theorem 4 (Kolmogorov's 0-1)

Let $\mathbf{X} = (X_1, X_2, \dots)$ *be independent. Any event stated in terms of the sequence* $(X_1, X_2, \dots)$*, not affected by the value of any of the* X_n*'s, has probability* 0 *or* 1*.*

Example

- Sounds weird?
- All examples from the last slide are of this type!
- In particular, if $\mathbf{X} = (X_1, X_2, \dots)$ are independent, then the series

$$\sum_{n=1}^{\infty} \frac{X_n}{b_n}$$

  either converges a.s. or diverges a.s. Whichever alternative holds? We have a theorem for that too.

How to prove? Show that any such event is independent of itself.

3 Series

Theorem 5 (Kolmogorov's Three Series Theorem)

Let $\mathbf{Y} = (Y_1, Y_2, \dots)$ *be independent. Let*

$$Z_n = \begin{cases} Y_n & |Y_n| \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

Then the series $\displaystyle\sum_{n=1}^{\infty} Y_n$ *converges a.s. if and only if all of the following conditions hold:*

1. $\sum_{n=1}^{\infty} P(|Y_n| > 1) < \infty$ *(large finitely otfen)*
2. $\sum_{n=1}^{\infty} E[Z_n] < \infty$ *(expectation of partial sums)*
3. $\sum_{n=1}^{\infty} E[(Z_n - E[Z_n])^2] < \infty$ *(variance of partial sums)*

Application

Consider the series $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{H_n}}{n}}_{=Y_n}$.

- $|Y_n| \le 1 \Rightarrow Z_n = Y_n$ and 1 ✓
- $E[Z_n] = 0, \Rightarrow 2$ ✓
- $E[Z_n^2] = \frac{1}{n^2} \Rightarrow 3$ ✓

Conclusion: converges a.s.

Other proofs? This Math Stack Exchange post.

Generalization

Random p-harmonic

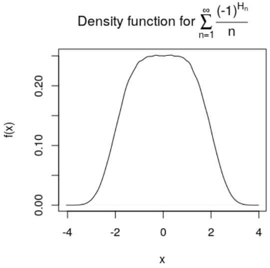
- Reminder: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges for all $p > 0$ (Theorem 3).
- What about
$$\sum_{n=1}^{\infty} \frac{(-1)^{H_n}}{n^p} \qquad (*)$$
- Conditions 1,2 in Theorem 5 trivially hold, with $Z_n = Y_n$.
- Check condition 3: $E[Z_n^2] = \frac{1}{n^{2p}}$.

Corollary 1

$$(*) \textit{ converges a.s. if } p > \frac{1}{2}, (*) \textit{ diverges a.s. if } p \leq \frac{1}{2}.$$

Simulations

Let's look at simulations for the random harmonic series.



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- More on the distribution? Read Byron Schmuland, Random Harmonic Series

What about other values of p?

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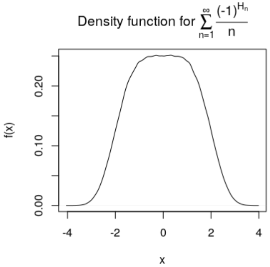

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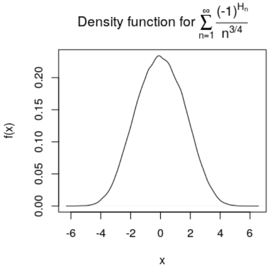

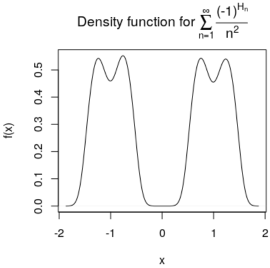

Random L-functions[1], 1/2

Construction

- Let $H_1 = 1$.
- For prime p, define H_p as before
- Extend to all natural numbers through the formula $H_{nm} = H_n + H_m$ (you can do this $\mod 2$).
- Example: $H_{p^n} = nH_p$, $H_6 = H_2 + H_3$, etc. Define

$$L(s) = \sum_{n=1}^{\infty} \frac{(-1)^{H_n}}{n^s} \qquad (**)$$

Almost the same as $(*)$, but here H_n are not independent! H_2 determines H_{2^n}, etc.

Corollary 2

$$(**) \text{ converges a.s. if } s > \frac{1}{2}.$$

[1] From Robert Hugh's lecture

Random L-functions, 2/2

Proof of Corollary 2

Key idea: bring this to the form of Theorem 5.

- By prime factorization,

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\text{ prime}}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right)=\prod_{p\text{ prime}}\left(1-\frac{1}{p^s}\right)^{-1},$$

Side note: can you see why $\sum_{p\text{ prime}}\frac{1}{p^s}$ converges if and only if $s>1$?

Random L-functions, 2/2

Proof of Corollary 2

Key idea: bring this to the form of Theorem 5.

- Because $n \to (-1)^{H_n}$ is multiplicative,

$$\sum_{n=1}^{\infty} \frac{(-1)^{H_n}}{n^s} = \prod_{p \text{ prime}} (1 + \frac{(-1)^{H_p}}{p^s} + \frac{(-1)^{2H_p}}{p^{2s}} + \dots)$$

$$= \prod_{p \text{ prime}} (1 - \frac{(-1)^{H_p}}{p^s})^{-1}$$

$$= \exp\left(- \sum_{p \text{ prime}} \ln(1 - \frac{(-1)^{H_p}}{p^s})\right)$$

Random L-functions, 2/2

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$$= \exp\left(- \sum_{p \text{ prime}} \ln(1 - \frac{(-1)^{H_p}}{p^s}) \right)$$

- Use Taylor expansion $-\ln(1-x) = x + x^2/2 + \dots$, to recover

$$\sum_{p \text{ prime}} \underbrace{\frac{(-1)^{H_p}}{p^s}}_{(I)} + \underbrace{\frac{1}{2p^{2s}} + \dots}_{(II)}$$

Random L-functions, 2/2

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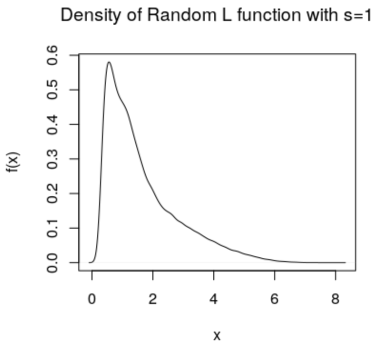
- So... when $s > \frac{1}{2}$
  - (II) converges (we mentioned earlier this slide).
  - (I) converges a.s., similarly to Corollary 1. $\square$

Simulations

You're probably curios, so here it is.



Discussion

- Very different from the distribution of the random harmonic series.
- Why positive?

Strong Law of Large Numbers

With the aid of the all-mighty Kronecker's Lemma one can use Theorem 5 to give an easy proof the SLLN, generalizations, and analogous results.

Theorem 6 (Strong Law of Large Numbers)

Let $X_1, X_2, \dots$ be independent and identically distributed with finite expectation μ. Let $S_n = X_1 + \cdots + X_n$. Then

$$\lim_{n\to\infty} \frac{S_n}{n} \to \mu \ a.s.$$

Discussion

In `MATH3160` we usually cover the Weak Law of Large Numbers:

- The WLLN claims the the difference between the empirical mean S_n/n and μ is "large" with asymptotically vanishing probability:

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0.$$

  There is **no** statement on actual convergence of the empirical means.
- The proof you usually see is based on Chebychev's inequality and assumes finite second moment.

Proof of Theorem 6

Lemma 7 (Kronecker's Lemma: summation by parts)

Suppose that

- $0 < a_1 < a_2 < \dots$ *with* $\lim_{n\to\infty} a_n = \infty$; *and*
- $\sum_{n=1}^{\infty} \frac{x_n}{a_n}$ *converges.*

then

$$\lim_{N\to\infty} \frac{1}{a_N} \sum_{n=1}^{N} x_n = 0.$$

Now for the proof.

- WLOG, assume $\mu = 0$.
- Apply Theorem 5 to the series $Y_n = X_n/n$ to conclude that $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s.
- Apply Kronecker's lemma with $x_n = X_n$ and $a_n = n$, to conclude that

$$\frac{S_N}{N} = \frac{\sum_{n=1}^{N} X_n}{N} \to 0 \text{ a.s.}$$

□

Done. Thank you.