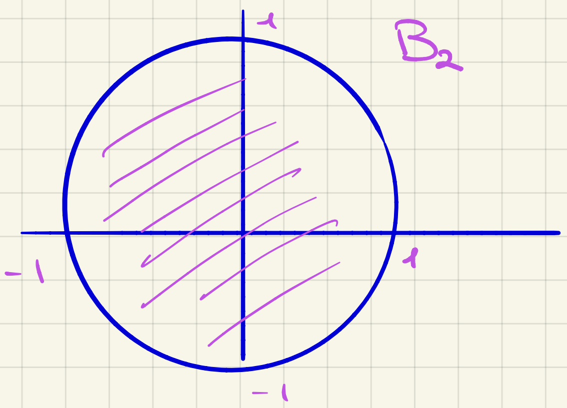
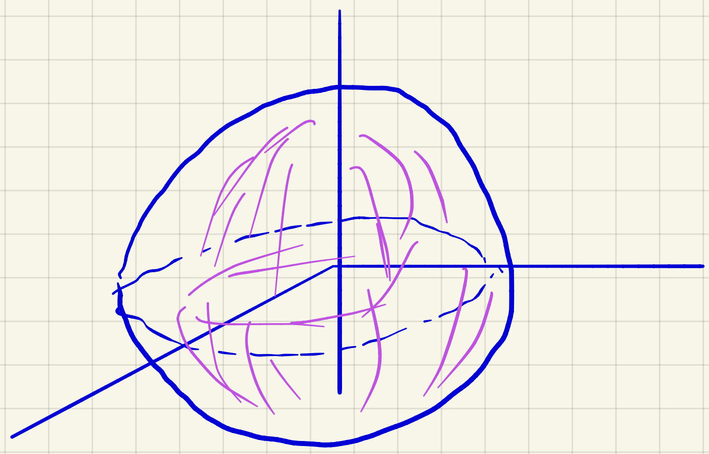
Volume of Balls

- We will consider balls of radius $r=1$ centered at the origin.

- in dimension $d=2$ $\quad B_2 = \{ \underbrace{(x,y)\in\mathbb{R}^2}_{\text{set of pts in the plane}} : x^2+y^2\le 1 \} \underbrace{\subseteq \mathbb{R}^2}_{\text{subset of the plane}}$



- $d=3$ $\quad B_3 = \{ (x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 1 \} \subseteq \mathbb{R}^3$



Something maybe you have never thought about: $d=1$?

- $d=1$ $\quad D_1 = \{ x\in\mathbb{R} : x^2\le 1 \} = [-1,1]$ $\quad$ 1-dimensional ball

- What's an n-dimensional ball?

it's the subset of $\mathbb{R}^n$ (n-dim space) defined as

$$B_n := \{ (x_1, \dots, x_n)\in\mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 \le 1 \}$$

<u>Q</u> What is the "size" (length/area/volume, a.k.a) of B_n?

- $n=1$ $\quad$ size$(B_1) = \ell([-1,1]) = 2$
- $n=2$ $\quad$ size$(B_2) = $ Area(Disk) $= \pi r^2 = \pi > 2 =$ size(B_1)
- $n=3$ $\quad$ size$(B_3) = $ Volume(Sphere) $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi > \pi =$ size(B_2)

Q: Does the size increase with n?

To answer this question, we should find an alternative way of computing size(B_2) and size(B_3).

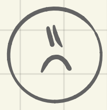
Q how can we find area B_2?

A calc 2 (or 3) if $A \subseteq \mathbb{R}^2$ is a subset of $\mathbb{R}^2$, then:

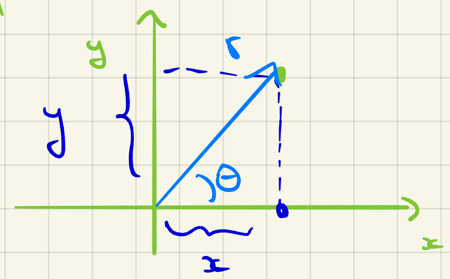
$$\text{Area}(A) = \iint_A 1\, dA$$

So if we wanna find Area(B_2) we should set-up a double integral over B_2.

we have 2 options

- cartesian coordinates:  messy
- polar coordinates:  nice

• polar coordinates



to find a point in the xy-plane you could either find (x,y), think of them as components of this vector, or (r,θ) think of them as angle vector/x-axis and length of the vector

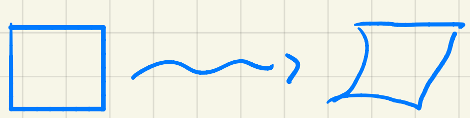
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

θ in $[0, 2\pi)$ $r \in [0,1]$ is a parametrization of B_2

$$\text{Area}(B_2) = \iint_{B_2} 1\, dA = \int_0^{2\pi}\int_0^1 1 \cdot \text{Jacobian}\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\, dr\, d\theta = 2\pi \cdot \frac{1}{2} r^2 \Big|_0^1 = \pi$$

polar coordinates: $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$

Jacobian = ? think about the map $(x,y) \longmapsto (\theta, \rho) = \left(\arctan\left(\frac{y}{x}\right); \sqrt{x^2+y^2}\right)$

small square 

it measures "how distorted is the "new square" compare to the original square"

Why you should appreciate this approach? it's easy to generalize!

$$\text{Volume}(B_3) = \iiint_{B_3} 1 \, dv \qquad \text{calc 3.}$$

B_3 is a 3-dim ball, we can parametrize it using spherical coordinates!

$$\begin{cases} x = r\cos\theta \sin\phi \\ y = r\sin\theta \sin\phi \\ z = r\cos\theta \end{cases} \qquad r \in [0,1], \quad \theta \in [0, 2\pi), \quad \phi \in [0, \pi]$$

is a parametrization of the unit sphere B_3

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2 \sin\phi$$

$$\text{Vol}(B_3) = \iiint_{B_3} 1 \, dv = \int_0^1 \int_0^{2\pi} \int_0^{\pi} 1 \cdot \text{Jacob} \; d\phi \, d\theta \, dr = \int_0^1 \int_0^{2\pi} -r^2 \cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta \, dr =$$

$$= \int_0^1 \int_0^{2\pi} 2r^2 \, d\theta \, dr = 4\pi \cdot \frac{1}{3} r^3 \Big|_0^1 = \frac{4}{3}\pi$$

Back to our original Q: size (B_n)?

these 2 examples suggest us to: parametrize B_n; set-up and evaluate an integral.

B_2:

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1 \end{cases}$$

B_3

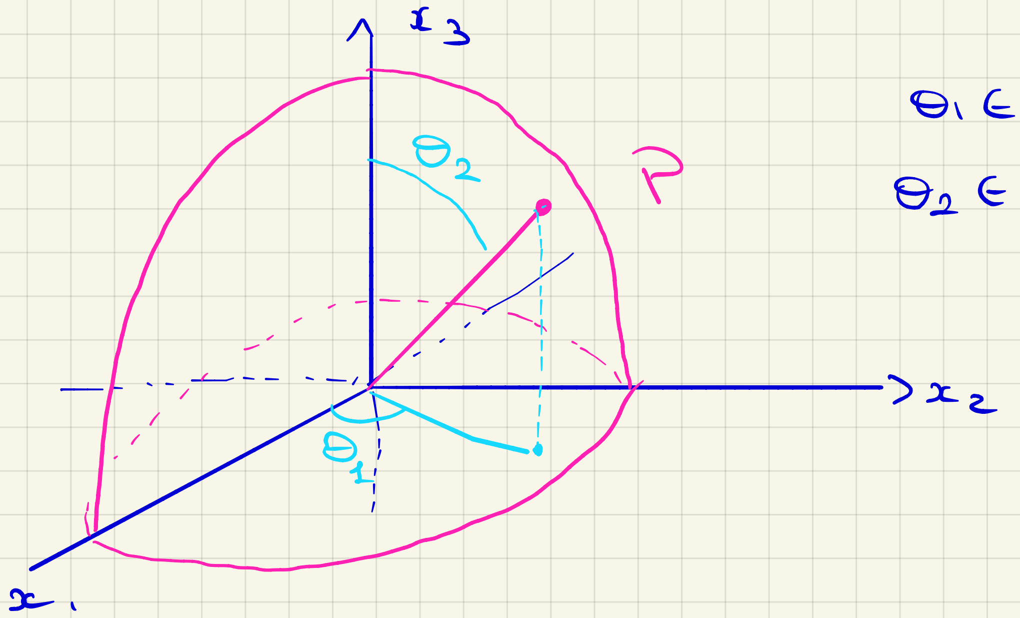
$$\begin{cases} x_1 = r\cos\theta_1 \sin\theta_2 \\ x_2 = r\sin\theta_1 \sin\theta_2 \\ x_3 = r\cos\theta_2 \end{cases}$$

B_4

$$\begin{cases} x_1 = r\cos\theta_1 \sin\theta_2 \sin\theta_3 \\ x_2 = r\sin\theta_1 \sin\theta_2 \sin\theta_3 \\ x_3 = r\cos\theta_2 \sin\theta_3 \\ x_4 = r\cos\theta_3 \end{cases}$$

Some comments about the meaning of these angles

$$\begin{cases} x_1 = r\cos\theta_1 \sin\theta_2 \\ x_2 = r\sin\theta_1 \sin\theta_2 \\ x_3 = r\cos\theta_2 \end{cases}$$

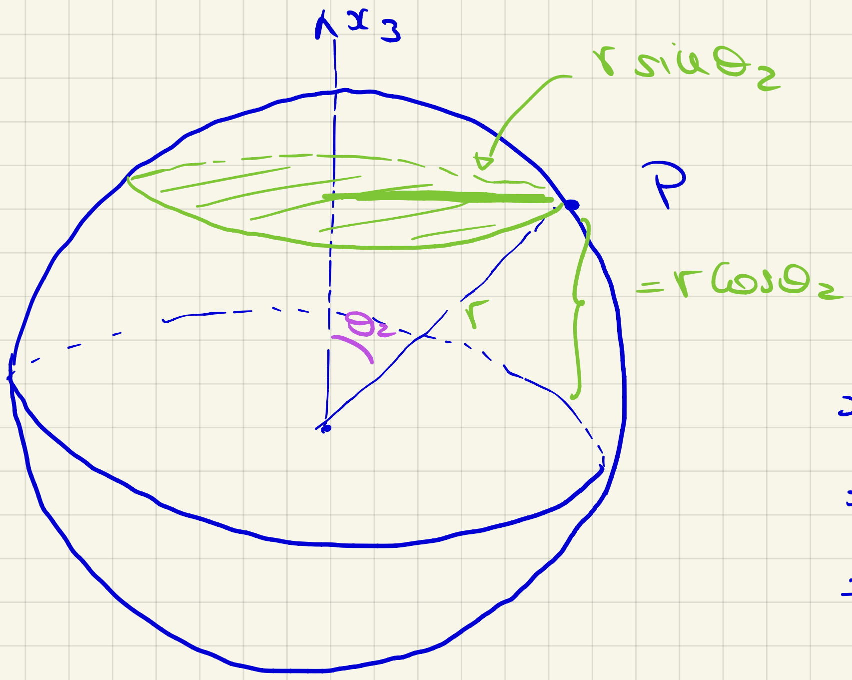


$\theta_1 \in [0, 2\pi)$

$\theta_2 \in [0, \pi]$

θ_2 $(=\phi)$ tells you how far you are from the x_3 $(=z)$ axis.

for a fix θ_2, we basically have a circle in the $x_3 = r\cos\theta_2$ plane of radius $r\sin\theta_2$



we can use polar coordinates to parametrize this circle:

$x_1 = (r\sin\theta_2)\cos\theta_1$

$x_2 = (r\sin\theta_2)\sin\theta_1$

$x_3 = r\cos\theta_2$

which are precisely the spher. coord.

think of a 3-dim. ball as made of a lot of 2-dim. disks one on top of the other, with radius changing according to the height.

. comment so it makes sense that $\theta \in [0, 2\pi)$, while $\phi \in [0, \pi]$

How should we think about the parametrization of B_4?

θ_3 says how far we are from the x_4-axis.

At height $r\cos\theta_3$, instead of a disk of radius $r\sin\theta_3$, you now have a sphere of radius $r\sin\theta_3$.

think of B_4 as a lot of 3-dim balls one on top of the other, with radius changing.

So a way of parametrizing B_n is

$$\begin{cases} x_1 = r\cos\theta_1 \sin\theta_2 \cdots\cdots \sin\theta_{n-1} & \theta_1 \in [0, 2\pi) \\ x_2 = r\sin\theta_1 \sin\theta_2 \cdots\cdots \sin\theta_{n-1} & \theta_j \in [0,\pi] \quad j = 2, \dots, n-1 \\ x_3 = r\cos\theta_2 \sin\theta_3 \cdots\cdots \sin\theta_{n-1} & r \in [0,1] \\ \vdots \\ x_{n-1} = r\cos\theta_{n-2}\sin\theta_{n-1} \\ x_n = r\cos\theta_{n-1} \end{cases}$$

what about the Jacobian?

polar (B_2) $J = r = r^{2-1} \cdot \sin\theta^{2-2}$ spherical (B_3) $J = r^2 \sin\phi = r^{3-1} \cdot \sin^{3-2}\theta_{3-1} \cdot \sin^{3-3}\theta_{3-2}$

so in general we have:

$$\text{Jacobian} = r^{n-1} \cdot \sin^{n-2}(\theta_{n-1}) \cdot \sin^{n-3}(\theta_{n-2}) \cdots \sin(\theta_2)$$

$$\text{size}(B_n) = \overbrace{\int \cdots \int_{B_n}}^{n\text{-times}} 1\, dv = \int_0^1 \underbrace{\int_0^{\pi} \cdots \int_0^{\pi}}_{(n-2)\text{-times}} \int_0^{2\pi} r^{n-1} \sin^{n-2}(\theta_{n-1}) \cdots \sin(\theta_2)\, d\theta_1 d\theta_2 \dots d\theta_{n-1} dr$$

$$= \int_0^1 r^{n-1} dr \cdot \int_0^{\pi} \sin^{n-2}(\theta_{n-1}) d\theta_{n-1} \cdots \int_0^{\pi} \sin(\theta_2) d\theta_2 \cdot \int_0^{2\pi} 1 \cdot d\theta_1$$

let us set $I_j = \int_0^{\pi} \sin^j(x)\, dx \qquad j \in \mathbb{N}_0$

We have: $\boxed{\text{size}(B_n) = \frac{2\pi}{n} \cdot I_1 \cdot I_2 \cdots I_{n-2}}$

Now we have to do some computations

integrate by parts twice

$$\cdot\ I_j = \int_0^{\pi} \sin^j(x)\, dx = \int_0^{\pi} \sin^{j-1}(x) \cdot \sin(x) dx = \frac{j-1}{j} \int_0^{\pi} \sin^{j-2}(x) dx$$

that is, $I_j = \frac{j-1}{j} I_{j-2} \qquad j \geq 2$

- $I_0 = \int_0^{\pi} \sin(x)^0 dx = \int_0^{\pi} 1\, dx = \pi$ - $I_1 = \int_0^{\pi} \sin x^1 dx = 2$

- $I_2 = \frac{2-1}{2} I_0 = \frac{1}{2} I_0 = \frac{1}{2}\pi$ $I_3 = \frac{3-1}{3} I_1 = \frac{4}{3}$

- $I_4 = \frac{4-1}{4} I_2 = \frac{3}{4} \frac{1}{2}\pi = \frac{3}{8}\pi$ $I_5 = \frac{5-1}{5} I_3 = \frac{4}{5} \cdot \frac{4}{3} = \frac{16}{15}$

So the volume of the higher dimensional unit ball is

$\text{Size}(B_3) = \frac{2\pi}{3} I_1 = \frac{4}{3}\pi$ $\text{Size}(B_4) = \frac{2\pi}{4} \cdot I_1 \cdot I_2 = \frac{\pi}{2} \cdot 2 \cdot \frac{1}{2}\pi = \frac{\pi^2}{2}$

$\text{Size}(B_5) = \frac{2\pi}{5} I_1 I_2 I_3 = \frac{2\pi}{5} \cdot 2 \cdot \frac{1}{2}\pi \frac{4}{3} = \frac{8}{15}\pi^2$

$\text{Size}(B_6) = \frac{2\pi}{6} I_1 \cdots I_4 = \frac{\pi}{3} \cdot \cancel{2} \cdot \frac{1}{\cancel{2}}\pi \cdot \frac{4}{\cancel{3}} \cdot \frac{\cancel{3}\pi}{8_2} = \frac{1}{6}\pi^3$

Of course one can write down a general formula for I_n.

$$I_{2k} = \frac{\pi (2k-1)!}{2^{2k-1} k! (k-1)!} \qquad I_{2k-1} = \frac{2^{2k-1} (k-1)!^2}{(2k-1)!}$$

For even-dimensional balls the formula is much nicer and is given by:

$$\text{Size}(B_{2n}) = \frac{\pi^n}{n!}$$

1^{st} punch line

- weird things happen, like $\lim_{n \to \infty} \text{Size}(B_{2n}) = \lim_{n\to\infty} \frac{\pi^n}{n!} = 0$

in higher dimensions, the ball "contains less volume".

the unit ball maximizes the volume when $n = 5$

→ in general, the dimension that maximizes the volume depends on the radius. ; e.g. $r = 2$ ⇒ maxim at $n = 24$

I honestly have no intuition why it happens at $n = 5$.

Bites of ∞-dimensional spaces

in $\mathbb{R}^n$ unit ball = set of points whose distance from the center is $= 1$.

$$B_R^{\mathbb{R}^n}(0) = \{x \in \mathbb{R}^n : d(x;0) \leq R\} = \text{ball}_{,\ rad = R,} \text{ centered at the origin}$$

$$B_R^{\mathbb{R}^n}(y) = \{x \in \mathbb{R}^n : d(x,y) \leq R\} = \text{ball}_{,\ rad = R,} \text{ centered at } y$$

of course $\underbrace{\text{size}(B_1^{\mathbb{R}^n}(0)) = \text{size}(B_1^{\mathbb{R}^n}(y))}_{(*)}$. $\Rightarrow \dfrac{\text{size}(B_1^{\mathbb{R}^n}(y))}{\text{size}(B_1^{\mathbb{R}^n}(0))} = 1$

2nd pencil line

(*) this is true only on $\mathbb{R}^n$ (well, on finite dimensional linear spaces)

Can you think about an ∞-dim. space?

$$W := \{ f : [0,1] \longrightarrow \mathbb{R} \text{ continuous and } f(0) = 0 \}$$

- $f, g \in W$, $a, b \in \mathbb{R} \Rightarrow af + bg \in W$
- $f \equiv 0 \in W$

eg. $f(x) = 4\sin x$
$f(x) = 1 - \cos x$

how can we define a distance between two "points" $f, g \in W$?

in $\mathbb{R}^2$: $d_{\mathbb{R}^2}(\underline{x};\underline{y}) = ((x_1-y_1)^2 + (x_2-y_2)^2)^{1/2}$

$d(f,g) := \max\limits_{0 \leq x \leq 1} |f(x) - g(x)|$ it's a distance.

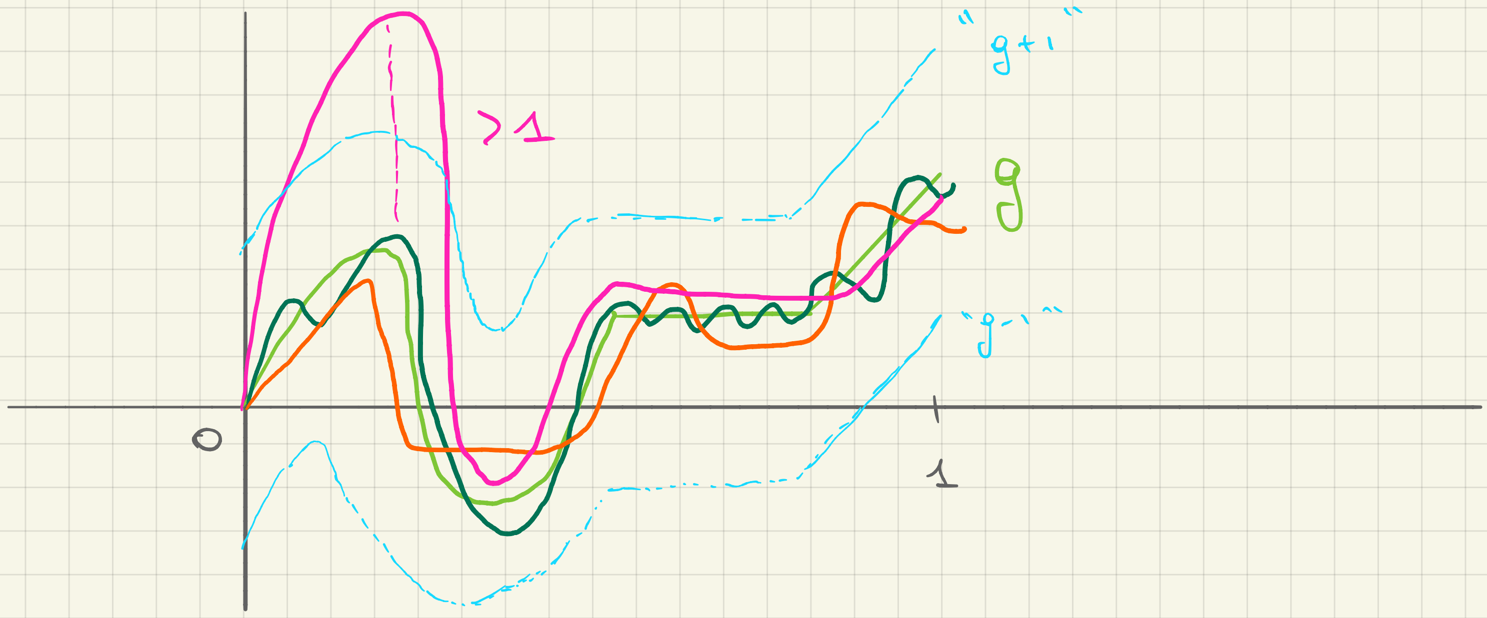
So now we can talk about balls in W:

$$B_1(0) = \{ f \in W : d(f,0) \leq 1\} = \{ f : [0,1] \to \mathbb{R} \text{ cont. } f(0) = 0 \ \max_{0 \leq x \leq 1} |f(x)| \leq 1 \}$$

unit ball centered at the "origin", aka zero-function.

$B_1(g) = \left\{ f \in W : \max_{0 \le x \le 1} |f(x) - g(x)| \le 1 \right\}$ unit ball centered at g

Can we visualize "points" in $B_1(g)$?



a "point" in $B_1(g)$ is a continuous function f with $f(0)=0$ and whose graph is entirely between those 2 blue curves

is the pink curve in $B_1(g)$? nope.

· What about translation invariance?

Recall that in $\mathbb{R}^n$ we have:

$$\text{size}\left(B_1^{\mathbb{R}^n}(0)\right) = \text{size}\left(B_1^{\mathbb{R}^n}(y)\right).$$

Assume you have a way of measuring size of balls in W.

prop Assume that, $\forall R>0$, $\forall f, g \in W$

$$\text{size}(B_R(f)) = \text{size}(B_R(g)).$$

if $\text{size}(B_R(0)) < \infty \implies \text{size}(B_R(0)) = 0$

Moral Assume you have a "way of measuring" subsets in W. assume that the size of a set and its translation are the same.

then the size of a ball is either 0 or ∞!

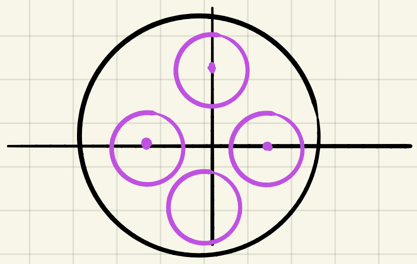
⇝ So if you want a non trivial way of measuring balls in W; you can't have translation invariance!!

proof give the idea. just draw something like ⊛

Since W is infinite dimensional, we can find ∞-many pairwise-disjoint balls contained in $B_R(0)$, say $B_r(f_n)$ for some $\{f_n\}_n \subset W$.

How is this possible?

in $\mathbb{R}^2$



⊛

the 4 purple balls all have the same radius and are centered on the x- and y-axis.

In ∞-dim. space, you might think like having ∞-many axis
⇒ so you have ∞-many disjoint balls contained in $B_R(0)$.

by our assumption $\text{size}(B_r(f_n)) = \text{size}(B_r(0)) =: c < \infty$ finite

each $B_r(f_n) \subseteq B_R(0)$ ⇒ so is the union

$$\bigcup_n B_r(f_n) \subset B_R(0)$$

$$\text{size}\left(\bigcup_n B_r(f_n)\right) \leq \text{size}(B_R(0)).$$

Q if two ball are disjoint, then

$$\text{size}(B_r(a_1) \cup B_r(a_2)) \overset{?}{=} \ldots \sum_{i=1}^{2} \text{size}(B_r(a_i))$$

$$\sum_{n=1}^{\infty} \text{size}(B_r(f_n)) \leq \text{size}(B_R(0)) < \infty$$

by assumption, all these sizes are the same

$$\sum_{n=1}^{\infty} c \leq \text{size}(B_R(0)) < \infty$$

Q is this possible? ⇝ yes: iff $c = 0$

· **Consequences:** since $\mathrm{size}(B_R(0)) \neq \mathrm{size}(B_R(f))$

the quotient is not 1, so one might ask what the value of

$$\frac{\mathrm{size}(B_R(f))}{\mathrm{size}(B_R(0))}$$ at least for small R.

$$\approx e^{-\frac{\lambda}{R^2}} \cdot e^{-\frac{1}{2}\int_0^1 (f')^2 dx}$$

if someone asks: λ is the smallest $\neq 0$ eigenvalue of $\frac{1}{2}\frac{\partial^2}{\partial x^2}$ on $[0,1]$.

· So, what do I study?

For keith: this might be too advance, but it could be interesting for juniors/ seniors

G = a Lie group

if you have taken Linear Algebra, you can think of G as being the set of orthogonal matrices, or upper-triangular-matrices. etc

· "algebra" / geometry

$W := \{ f : [0,1] \to G$ continuous $f(0) = 0 \}$

topology

· W is an ∞-dimensional space $\rightsquigarrow$ functional analysis

· how can you measure size of balls in W?

simplest way I know: Brownian Motion probability.

What do I like about what I do:

geometry / algebra $\longleftrightarrow$ analysis / probability

I really like when different fields of math combine with each other.