Tom Roby **Combinatorial Triangles** **29 January 2020**

Some sequences are some important that we recongize them instantly: squares, primes, triangular numbers, Fibonacci numbers, Catalan numbers, etc. Some of these are doubly-indexed, which makes them more easily described and understood as *combinatorial triangles*. We'll try look at about six (!) examples in less than 60 minutes.

1. These triangles typically represent *disaggregated* data, i.e., their row or column sums are frequently of interest.
2. These numbers frequently come up as coefficients of simple-to-define polynomials or infinite series.
3. Most have a core combinatorial "story" which can be used to give a nice explanation of a recursion.
4. Many examples come up all the time in applications;
5. Mathematicians frequently rediscover the same sequence in completely different contexts; the OEIS makes it simple to see whether a sequence that comes up in A's work is already known, and perhaps connect A's work to other interesting mathematics.

Pingala–Khayyam–YangHui–Pascal Triangle

Definition 1. *Let $\binom{n}{k}$ denote the number of k-element subsets of $[n] := \{1, 2, 3, \ldots, n\}$.*

This one has perhaps the most obvious recursion of all:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$
\begin{array}{ccccccccccccccccccc}
&&&&&&&&& 1 \\
&&&&&&&& 1 && 1 \\
&&&&&&& 1 && 2 && 1 \\
&&&&&& 1 && 3 && 3 && 1 \\
&&&&& 1 && 4 && 6 && 4 && 1 \\
&&&& 1 && 5 && 10 && 10 && 5 && 1 \\
&&& 1 && 6 && 15 && 20 && 15 && 6 && 1 \\
&& 1 && 7 && 21 && 35 && 35 && 21 && 7 && 1 \\
& 1 && 8 && 28 && 56 && 70 && 56 && 28 && 8 && 1 \\
1 && 9 && 36 && 84 && 126 && 126 && 84 && 36 && 9 && 1
\end{array}
$$

Stirling numbers for set partitions (2nd kind)

Definition 2. $\left\{ n \atop k \right\}$:= # *ways to partition* $[n]$ *into exactly* $[k]$ *(nonempty) blocks. Read "n subset k"*

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&0&&1&&&&&& \\
&&&&&0&&1&&1&&&&& \\
&&&&0&&1&&3&&1&&&& \\
&&&0&&1&&7&&6&&1&&& \\
&&0&&1&&15&&25&&10&&1&& \\
&0&&1&&31&&90&&65&&15&&1& \\
0&&1&&63&&301&&350&&140&&21&&1
\end{array}
$$

Stirling numbers for permutations (1st kind)

Definition 3. $\left[n \atop k \right]$:= # *ways to arrange* $[n]$ *into exactly* $[k]$ *cycles. Read "n cycle k".*

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&0&&1&&&&&& \\
&&&&&0&&1&&1&&&&& \\
&&&&0&&2&&3&&1&&&& \\
&&&0&&6&&11&&6&&1&&& \\
&&0&&24&&50&&35&&10&&1&& \\
&0&&120&&274&&225&&85&&15&&1& \\
0&&720&&1764&&1624&&735&&175&&21&&1
\end{array}
$$

Sideways Numbers

What happens if we change our recursion to add sideways (and carry from right end to left):

$$T(n,k) = T(n-1,k-1) + T(n,k-1)\,,$$

with $T(0,0) := 1$, and $T(n,0) := T(n-1,n-1)$?

$$\begin{array}{c}
1 \\
1 \quad 2 \\
2 \quad 3 \quad 5 \\
5 \quad 7 \quad 10 \quad 15 \\
15 \quad 20 \quad 27 \quad 37 \quad 52 \\
52 \quad 67 \quad 87 \quad 114 \quad 151 \quad 203 \\
203 \quad 255 \quad 322 \quad 409 \quad 523 \quad 674 \quad 877 \\
877 \quad 1080 \quad 1335 \quad 1657 \quad 2066 \quad 2589 \quad 3263 \quad 4140
\end{array}$$

Eulerian Numbers

Definition 4. *An* ***ascent*** *of a permutation $w = a_1 \cdots a_n$ is a value $j \in [n-1]$ with $a_j < a_{j+1}$. $\left\langle {n \atop k} \right\rangle :=$ # permutations of $[n]$ with exactly k ascents.*

$$\begin{array}{c}
1 \\
1 \quad 0 \\
1 \quad 1 \quad 0 \\
1 \quad 4 \quad 1 \quad 0 \\
1 \quad 11 \quad 11 \quad 1 \quad 0 \\
1 \quad 26 \quad 66 \quad 26 \quad 1 \quad 0 \\
1 \quad 57 \quad 302 \quad 302 \quad 57 \quad 1 \quad 0 \\
1 \quad 120 \quad 1191 \quad 2416 \quad 1191 \quad 120 \quad 1 \quad 0
\end{array}$$

Alternating Permutations

Definition 5. *Call a permutation $w = a_1 \cdots a_n$* ***alternating*** *if it satisfies $a_1 > a_2 < a_3 > a_4 < \cdots$. ($w$ with all inequalities reversed are called* ***reverse alternating****.)*

$$E_n := \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\}.$$

EG: $E_4 = 5$, counting $2143, 3142, 3241, 4132, 4231$.

Definition 6. *Let $E_{n,k}$ denote number of alternating permutations in $\mathfrak{S}_{n+1}$ with first term $k+1$, the* ***Entringer number****.*

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & & & & & \\
 & & & & 1 & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 & & & \\
 & & 5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & 16 & \\
61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 & \leftarrow & 0
\end{array}
$$

Theorem 7 (André, 1879)**.**

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x = 1 + 1x + 1\frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!} + 272\frac{x^7}{7!} + \dots$$

Key to proof is to show combinatorially that $2E_{n+1} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for $n \geq 1$.

References

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[Stan99] Richard P. Stanley, *Enumerative Combinatorics Volume 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.

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[WikiP] Wikipedia entries for *Pascal's Triangle* and links therein to individual mathematicians.