# Reverse Hex

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## Contents



**Abstract**

The game of Hex has been a popular subject of study since John Nash first began studying it in the 1940's. Since then, numerous variations of the game of Hex have been created. Misère Hex, or Reverse Hex, is a variation of the game of Hex where the goal of the game is to lose. In this paper, we determine that player 1 in Reverse Hex has a winning strategy if the board has even dimensions, and player 2 has a winning strategy if the board has odd dimensions. We will also briefly discuss two other variations of the game of Hex, Vex and Y, and how their properties are similar to those of Hex and Reverse Hex.

# 1 Introduction

The game of Hex traces its origins back to 1942 when it was discovered by Danish mathematician Piet Hein. It was later rediscovered in 1949 by John Nash. In 1953, the game of Hex was released by the Parkers Brothers and marketed to the public. In 1957, Martin Gardner discussed the game of Hex in a column in *Scientific American*, further popularizing the game as a subject of study among mathematicians. Today, research regarding the game of Hex is mostly spent in developing programs that can play the game of Hex perfectly, which has had mixed results ([PSSW05]).

The rules of the Hex are simple. Two players take turns filling placing stones into cells, with the objective of constructing a path connecting two opposite sides. The rules will be explained in more depth in Section 2.2.

The game of Hex has major consequences in mathematics. It has been shown that the game of Hex cannot end in a draw. This is known as the Hex theorem, and will be more formally presented and explained elsewhere in this paper. More importantly, it has been shown that the Hex theorem is equivalent to the Brouwer Fixed-Point theorem, an important theorem in the study of topology. This has been shown in [Gal79].

The game of Hex has inspired numerous variations that have also been studied. In particular, this paper focuses on Reverse Hex, also known as Misère Hex, or Rex. In Reverse Hex, the goal is essentially to lose the game of Hex. These rules will be further explained in Section 2.3.

The goal of this paper is to present a proof of who has a winning strategy in Reverse Hex. More specifically, we will prove the following theorem taken from [LS99].

**Theorem 1.1.** *In Reverse Hex, player 1 has a winning strategy if the dimensions of the board are even, and player 2 has a winning strategy if the dimensions of the board are odd. Also, the losing player has a strategy that forces all of the cells to be filled before the other player can win.*

To further support this proof, we will also include a proof of the Hex theorem.

**Theorem 1.2.** *The game of Hex cannot end in a draw.*

However, we will slightly modify the proof so that its results more specifically discuss Reverse Hex. In other words, we will actually prove the following theorem.

**Theorem 1.3.** *The game of Reverse Hex cannot end in a draw.*

This paper begins by providing background information in Section 2. Within this background section, in Section 2.1, we will provide a basic introduction into some game theory terminology that will be used throughout this paper. In Section 2.2, we will provide a more in depth description of the rules of Hex. In Section 2.3, we will provide a more in depth description of the rules of Reverse Hex and describe the monotonicity property of Reverse Hex that will be instrumental in proving our main theorem. In Section 2.4, we will provide some background in graph theory that is necessary for our proof of Theorem 1.3. In Section 3, we will present three proofs. In Section 3.1, we will first prove a graphing lemma that will assist us in our proofs. We will then prove Theorem 1.3 using our modified proof of the Hex theorem. In Section 3.2, we will prove our main result, Theorem 1.1. In Section 4, we briefly discuss other variations of the game of Hex and their properties. This consists of the game of Vex in Section 4.1, and Y in Section 4.2. We then conclude with some statements regarding possible future research into the subject.

# 2 Background and Definitions

Before we can prove who has a winning strategy in Reverse Hex, we must first provide some background. We will start by providing a basic introduction into game theory and its terminology. We will then provide some background regarding the game of Hex as it will make understanding Reverse Hex easier. We will then provide some background on Reverse Hex. Lastly, we will provide a basic introduction into some graph theory terminology that will be used in our proofs.

## 2.1 Game Theory

We will start with a brief introduction to some game theory terminology that will be used throughout the paper.

**Definition 2.1.** (From [SG12]) A *strategy* is a player's plan of action for any and every situation that player might encounter.

Throughout this paper, the term winning strategy will be used often.

**Definition 2.2.** A *winning strategy* is a strategy that is guaranteed to result in the player winning regardless of his or her opponent's actions.

## 2.2 Hex

We will now provide an intuitive explanation of the rule of Hex.

**Definition 2.3.** ([Mil02]) The game *Hex* is a two player game played on an $n \times n$ board of hexagonal cells, as shown in Figure 1. Each player is assigned two opposite sides, and the players take turns placing stones on the cells until one has constructed a path connecting their two sides.

One of the most important results about the game of Hex is given by Theorem 1.2, often referred to as the Hex Theorem. This theorem is important, as its proof will form the foundation for our proof that the game of Reverse Hex cannot end in a draw (Theorem 1.3).

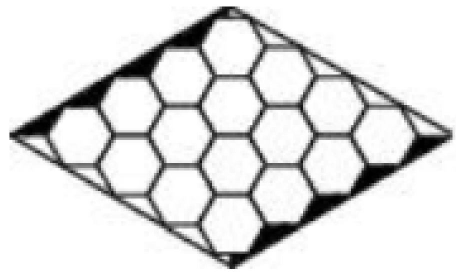

**Figure 1:** Hex board (Part of Figure 4.1 in [PSSW05])

This theorem is also equivalent to the Brouwer Fixed-Point Theorem, as shown in [Gal79]. It has already been determined that the first player in Hex always has a winning strategy ([Mil02]).

## 2.3 Reverse Hex

We will now provide an intuitive explanation of the rules of Reverse Hex.

**Definition 2.4.** ([HTH12]) *Reverse Hex* is a variation of the game of Hex where the rules are exactly the same, except the goal is to not construct a path connecting your two sides. In other words, whoever is forced to complete a path connecting their two sides loses.

Reverse Hex has a very unique characteristic that is essential for proving our main theorem.

**Definition 2.5.** ([LS99]) Suppose in a game of Reverse Hex there is a layout that is a win for player 1. By definition there must a path of player 2's stones connecting his two sides. Now suppose that a subset of the empty cells were randomly filled in with either player 1's stones or player 2's stones. This layout would still be a win for player 1, as it would not affect player 2's path. This is the *monotonicity property* of Reverse Hex.

## 2.4 Graph Theory

Given the overall graphical nature of Reverse Hex, our proofs will involve some graph theory. Thus it is necessary to provide some background here. We begin with the definition of a graph.

**Definition 2.6.** ([Ruo13]) A graph consists of a set of points called *vertices* and a collection of *edges* that either connecting two vertices or connect a vertex to itself.

Given a graph, it is often useful to record information about the number of edges emanating from the vertices of the graph.

**Definition 2.7.** ([Ruo13]) The *degree* of a vertex is the number of edges for which the vertex is an end vertex.

As a special case, a graph may contain vertices with no edges emanating from them.

**Definition 2.8.** ([Ruo13]) An *isolated point*, or *isolated vertex*, is a vertex with a degree of 0.

It is often useful to study the structure of a graph by looking at certain subsets of the graph.

**Definition 2.9.** ([gra]) A *path* is a series of vertices in a graph where every adjacent pair of vertices are connected by an edge of the graph. A path that does not repeat vertices is called a *simple path*. A path that begins and ends at the same vertex is called a *circuit*. A circuit that does not repeat vertices is called a *simple cycle*.

# 3 Proofs of Theorem 1.1 and Theorem 1.3

We will prove Theorem 1.1. In order to do this, there are actually two things we must prove. First, we will prove Theorem 1.3 which states that Reverse Hex cannot end in a draw. We will do this by utilizing a modified version of a proof of the Hex theorem (Theorem 1.2). Second, we will prove that the player who does not have a winning strategy has a strategy that forces every cell to be filled before he loses. We will then show that our Theorem 1.1 flows naturally from these results.

## 3.1 Why Reverse Hex Cannot End in a Draw

Before we can prove which player has a winning strategy in Reverse Hex, we must first prove that there must be a winner. In other words, we must prove Theorem 1.3, which state that the game of Reverse Hex cannot end in a draw. However, before we can prove this, we must first prove the following lemma from [Jin11].

**Lemma 3.1.** *A finite graph whose vertices have a degree of at most two must be the union of disjoint sub graphs that are either a simple cycle, a simple path, or an isolated point.*

The following proof is from [Jin11].

*Proof.* We will prove Lemma 3.1 by inducting on the number of edges in the graph. We will use $G_k$ to denote a graph with $k$ edges. Note that since there are no vertices with degree greater than 2, the number of edges must be less than or equal to the number of vertices.

In the base case where there are 0 edges, all vertices will be isolated points, and thus our lemma holds. We will now assume that $G_n$ is the union of disjoint sub graphs that are either a simple cycle, a simple path, or an isolated point. For $G_{n+1}$, we will remove an edge that connects two arbitrary vertices $u$ and $v$. Because $u$ and $v$ had at most a degree of 2, they now

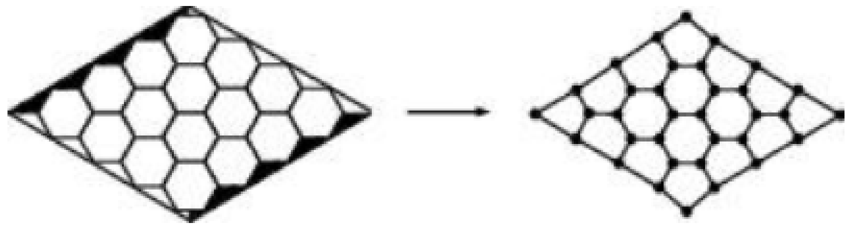

**Figure 2:** Hex board as a planar graph (Figure 4.1 in [PSSW05])

have at most a degree of 1. Thus, they cannot be part of any cycles. By our assumption, $G_n$ consists of disjoint simple cycles, disjoint paths, and isolated points. If we add back the edge connecting vertices $u$ and $v$, all sub graphs disjoint from $u$ and $v$ are unchanged, and those connected to $u$ or $v$ are now part of either a simple path or a simple cycle. Thus, $G_{n+1}$ is the union of disjoint sub graphs that are either a simple cycle, a simple path, or an isolated point. Thus, our lemma holds for all graphs $G_k$ such that $k \geq 0$.

□

With Lemma 3.1 proven, we can now proceed to prove Theorem 1.3. This proof is an modified version of the proof of Theorem 1.2 from [PSSW05] altered so that it applies to the game of Reverse Hex.

*Proof.* There are two different cases that would result in a draw, both players winning, and both players losing. However, it is impossible for both players to lose because the game ends as soon as one player completes a path, so we only need to worry about the case where both players win. We will show that both players cannot win by showing that one player must complete a path connecting their two sides.

Think of the Hex board as a planar graph, as shown in Figure 2. Now consider a completed game of Reverse Hex in which all the spaces are filled. We will create a graph $G$ by coloring all the edges that lie between two differently filled cells. Each vertex on $G$ will have a degree of either 0 or 2, with the exception of the four corners, which will all have a degree of 1. Because no vertex has a degree greater than 2, we know that by Lemma 3.1 $G$ is the union of disjoint simple cycles, simple paths, and isolated points. Because there are exactly 4 vertices of degree 1, we will have exactly two disjoint simple paths connecting the corners. Because they are disjointed, each path must connect either the top or bottom corner to the left or right corner, and each path must connect different corners. This will gives us a graph that looks like Figure 3.

Since there must always be a path connecting the top or bottom to one of the sides, and all the tiles on one side of the paths must be the same color, there must always be a losing path connecting two sides. Thus, it is impossible for both players to win Reverse Hex. Therefore, Reverse Hex cannot end in a draw.

□

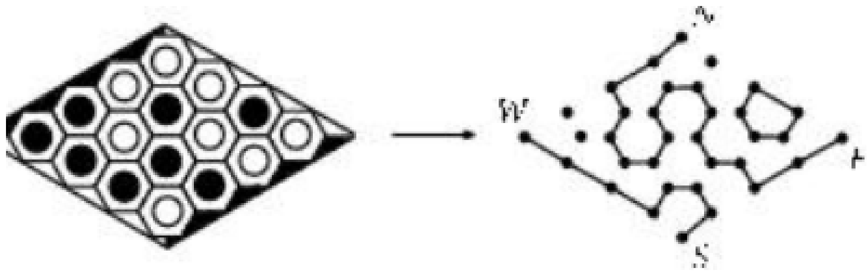


**Figure 3:** Sample graph $G$ (Figure 4.3 in [PSSW05])

## 3.2 Who Has the Winning Strategy in Reverse Hex?

We will now prove the Theorem 1.1. We will use the proof from [LS99] to support this theorem.

*Proof.* We will first prove the second part of theorem 1.1, that the losing player has a strategy that forces all of the cells to be filled before the other player can win. We will do this by showing that the minimum number of cells left unfilled for a winning strategy is 0.

First, let W denote the player with a winning strategy $\mathcal{L}$, and L denote the player without a winning strategy. Also denote $m(\mathcal{L})$ to be the minimum number of cells left empty by $\mathcal{L}$. Assume that $m(\mathcal{L}) \geq 1$. We will show that this contradicts our assumption that L does not have a winning strategy. We will need to show this in two different cases, when L is the first player, and when L is the second player.

When L is the first player, they can apply an arbitrary first move. They will then create an imaginary game, where the cell they just played in is still empty. Thus, the imaginary game resembles the real game exactly, except the cell that was arbitrarily played in is considered empty in the imaginary game, while it has L's stone in it in the real game. L will have a winning strategy $\mathcal{L}^r$ for the imaginary game because they can steal the strategy that W has for the real game When $\mathcal{L}^r$ requires L to play in the cell that is filled in the real game, L instead arbitrarily fills a cell and creates a new imaginary game where that cell is considered empty. Due to the fact that $m(\mathcal{L}^r) \geq 1$, W will always be able to play because there will be at least two empty cells in the imaginary game, and at least one empty cell in the real game. This also means that L will be able to play on his or her turn because there will be at least three empty cells in the imaginary game, and at least two empty cells in the real game. Therefore, the real game will continue as long as the imaginary game continues.

Because $\mathcal{L}^r$ is a winning strategy for L in the imaginary game, L will win the imaginary game. Due to the monotonicity property of this game as explained in Definition 2.5, L will also have won the real game, thus contradicting our assumption that W had the winning strategy and L did not.

We will now look at the scenario where L is the second player. In this case, W plays the first move. L then creates an imaginary game where the cell that W played in is considered empty, and employs the winning strategy $\mathcal{L}^r$ for the imaginary game. Similar to the first case, whenever $\mathcal{L}^r$ requires L to play in the cell that is filled in the real game, L instead plays in an arbitrary cell, and creates a new imaginary game where that cell is considered empty. From here on out, it is the same as the first case. Because $m(\mathcal{L}^r) \geq 1$, the real game will continue as long as the imaginary game continues, and because $\mathcal{L}^r$ is a winning strategy for the imaginary game, L will win the imaginary game. Again, due to the monotonicity property of Reverse Hex, L will win the real game, contradicting our assumption that W had a winning strategy and L did not.

Thus, we have proven that $m(\mathcal{L}) = 0$. From this, we can determine the winner simply by the dimensions of the board due to the fact that whoever makes the last move must be the loser and that there cannot be a draw. Therefore, it follows that on a board with even dimensions, and thus an even number of cells, player 1 has the winning strategy because in optimal play, player 2 will be forced to make the last move. Similarly, on a board with odd dimensions, player 2 has the winning strategy because in optimal play, player 1 will be forced to make the last move.

$\square$

While this proof is sufficient to prove Theorem 1.1, it has been further strengthened in [HTH12] by utilizing results about variations of Reverse Hex called punctured Rex and terminated punctured Rex. However, for the sake of brevity and simplicity, these proofs were left out.

# 4 Other Variants of Hex

Lastly, we will end with a discussion of other variants of Hex, and how different properties of Hex (including the Hex theorem) can extend or relate to these different variants.

## 4.1 Vex

Vex, like Reverse Hex, is a variation of Hex where the objective of the game is modified.

**Definition 4.1.** (From [PSSW05]) *Vex* is played very similarly to Hex, except the first player starts in one of the obtuse corners and wins if they create a path connecting his or her starting piece to one of the two opposite sides. The second player wins if they prevent the first player from completing a path.

Vex, like Hex, cannot end in a draw. This follows naturally from the objectives of the game (if one player loses, it implies the other player won, and vice versa). It can also be shown that like Hex, the first player has a winning strategy in Vex. This follows from Piet Hein's observation that both players cannot be blocked locally ([PSSW05]).

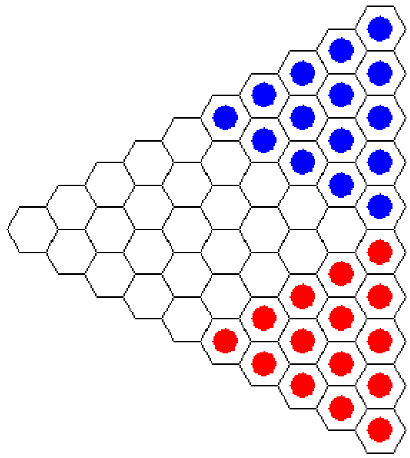

**Figure 4:** Hex as a special case of Y (From [Bog])

### 4.2 Y

Another variation of the game of Hex is simply referred to as Y. Unlike Reverse Hex and Vex, Y not only changes the objective of the game, but also the board that the game is played on.

**Definition 4.2.** From [PSSW05]) *Y* is played using the same rules as Hex except it is played on a triangular board, and the goal of each player is to create a single path that connects all three sides.

While there is currently a proof showing that Y, like Hex, cannot end in a draw, this proof is considered to be flawed. However, Jack van Rijswijck, the author of the proof, has recently refined the proof in his PH.D. thesis. This corrected proof is currently pending publication ([Bog]).

One interesting fact about Y is that Hex is actually a special case of Y where each player has completely filled a corner of the board (as illustrated in Figure 4). Thus, to show that Y cannot end in a draw would also prove that the game of Hex cannot end in a draw, and thus is an alternative approach to proving the Hex theorem ([Bog]).

## 5 Conclusion

In this paper, we have proven that Reverse Hex cannot end in a draw. More importantly, we have proven that the first player has the winning strategy when the board has even dimensions, and that the second player has the winning strategy when the board has odd dimensions. We have also shown that the player without a winning strategy has a strategy that forces every cell to be filled before the game's conclusion. We then concluded by briefly touching upon other variations of the game of hex that have similar properties to the game of Hex and Reverse Hex.

Though we have proven that there is always a winner in Reverse Hex and who has the winning strategy in any given game of Reverse Hex, there is still much more to research. In [HTH12], they prove what some winning moves are for player 1 on boards with even dimensions, and what winning counter moves are for player 2 on boards with odd dimensions in Reverse Hex. However, there may be more winning moves that have not yet been proven. Thus, there is further room for research.

There are also numerous variations of Hex besides Reverse Hex and the ones we briefly discussed in Section 4 that have not received as much attention. These variations not only vary the objective of the game and the shape of the board, but also the actual method of playing. A good, but not exclusive list of variations that could and should receive further attention is included in [PSSW05].

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