Reading Assignment 3 (submit at the end of class, Tues, Feb 25, 2020)

> Use the introduction to "sell" the key points of your paper.
>
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1 How to write an introduction, according to [Kle05].

The introduction is where readers settle into the "story," and often make the final decision about reading the whole paper. Start strong; don't waste words or time. Your readers have just read your title and abstract, and they have gained a general idea of your subject and treatment. However, they are probably still wondering what exactly your subject is and how you'll present it. A strong introduction answers these questions with clarity and precision, but in nontechnical terms. It identifies the subject precisely, and instills interest in it by giving details that did not fit into the title or abstract, such as how the subject arose and where it is headed, how it relates to other subjects and why it is important. A strong introduction touches on all the significant points, and no more. A strong introduction gives enough background material for understanding the paper as a whole, and no more. Put background material pertinent to a particular section in that section, weaving it unobtrusively into the text. A strong introduction discusses the relevant literature.

Finally, a strong introduction describes the organization of the paper, making explicit references to the section numbers. It summarizes the contents in more detail than the abstract, and it says what can be found in each section. It gives a road map, which indicates the route to be followed and the prominent features along the way. This road map is essentially a table of contents in a paragraph of prose. It is always placed at the end of the introduction to ease the transition into the next section.

What are some things that should be part of a strong introduction?

2 Sample Introductions Activity

Read each of the following sample introductions and write some comments about each one. Think about how well each sample introduction addresses the requirements listed in Section 1.

2.1 Sample Introduction (algebraic and transcendental numbers)

Abstract algebra is a field that results from stripping away artificial structures we impose on our own number systems, and building up from basic concepts. Ultimately, we reduce everything to sets with operations on them, and yet this can give us surprisingly deep and fundamental results in every area of mathematics.

This paper is organized as follows. In Section 2, we will discuss how algebra can be used to organize all numbers into two categories: algebraic and transcendental. In Section 3, we will place the famous constants e and π in the latter, and then prove the impossibility of the ancient question of squaring the circle.

Comments about the sample introduction

2.2 Sample Introduction (game of Hex)

The game of Hex traces its origins back to 1942 when it was discovered by Danish mathematician Piet Hein. It was later rediscovered in 1949 by John Nash. In 1953, the game of Hex was released by the Parkers Brothers and marketed to the public. In 1957, Martin Gardner discussed the game of Hex in a column in *Scientific American*, further popularizing the game as a subject of study among mathematicians. Today, research regarding the game of Hex is mostly spent in developing computer programs that can play the game of Hex perfectly, which has had mixed results ([PSSW05]).

The rules of the Hex are simple. Two players take turns filling placing stones into cells, with the objective of constructing a path connecting two opposite sides. The rules will be explained in more depth in Section 2.2.

The game of Hex has major consequences in mathematics. It has been shown that the game of Hex cannot end in a draw. This is known as the Hex theorem, and we will formally present and explain this theorem in this paper. More importantly, it has been shown in [Gal79] that the Hex theorem is equivalent to the Brouwer Fixed-Point theorem, an important theorem in the study of topology.

The game of Hex has inspired numerous variations that have also been studied. In particular, this paper focuses on Reverse Hex, also known as Misère Hex, or Rex. In Reverse Hex, the goal is essentially to lose the game of Hex. These rules will be further explained in Section 2.3.

The main goal of this paper is to present a proof of who has a winning strategy in Reverse Hex. More specifically, we will prove the following theorem taken from [LS99].

Theorem 2.1. *In Reverse Hex, player 1 has a winning strategy if the dimensions of the board are even, and player 2 has a winning strategy if the dimensions of the board are odd. Also, the losing player has a strategy that forces all of the cells to be filled before the other player can win.*

To further support this proof, we will also include a proof of the Hex theorem.

Theorem 2.2. *The game of Hex cannot end in a draw.*

However, we will slightly modify the proof so that its results more specifically discuss Reverse Hex. In other words, we will actually prove the following theorem.

Theorem 2.3. *The game of Reverse Hex cannot end in a draw.*

This paper begins by providing background information in Section 2. Within this background section, in Section 2.1, we will provide a basic introduction into some game theory terminology that will be used throughout this paper. In Section 2.2, we will provide a more in depth description of the rules of Hex. In Section 2.3, we will provide a more in depth description of the rules of Reverse Hex and describe the monotonicity property of Reverse Hex that will be instrumental in proving our main theorem. In Section 2.4, we will provide some background in graph theory that is necessary for our proof of Theorem 2.3. In Section 3, we will present three proofs. In Section 3.1, we will first prove a graphing lemma that will assist us in our proofs. We will then prove Theorem 2.3 using our modified proof of the Hex theorem. In Section 3.2, we will prove our main result, Theorem 2.1. In Section 4, we briefly discuss other variations of the game of Hex and their properties. This consists of the game of Vex in Section 4.1, and Y in Section 4.2. We then conclude with a few open questions in the subject.

Comments about the sample introduction

2.3 Sample Introduction (Approximations of π)

Numerical estimates for π have been found in records of several ancient civilizations. These estimates were all based on inscribing and circumscribing regular polygons around a circle to get upper and lower bounds on the area (and thus upper and lower bounds on π after dividing the area by the square of the radius). Such estimates are accurate to a few decimal places. Around 1600, Ludolph van Ceulen gave an estimate for π to 35 decimal places. He spent many years of his life on this calculation, using a polygon with 2^{62} sides!

With the advent of calculus in the 17-th century, a new approach to the calculation of π became available: infinite series. For instance, if we integrate

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots, \qquad |t| < 1$$

from $t = 0$ to $t = x$ when $|x| < 1$, we find

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots. \tag{1}$$

Actually, this is also correct at the boundary point $x = 1$. Since $\arctan 1 = \pi/4$, the equality (1) specializes to the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots, \tag{2}$$

which is due to Leibniz. It expresses π in terms of an alternating sum of the reciprocals of the odd numbers. However, the series in (2) converges much too slowly to be of any numerical use. For example, truncating the series after 1000 terms and multiplying by 4 gives the approximation $\pi \approx 3.1405$, which is only good to two places after the decimal point.

There are other formulas for π in terms of arctan values, such as

$$\begin{aligned} \frac{\pi}{4} &= \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) \\ &= 2\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) \\ &= 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right). \end{aligned}$$

Since the series for $\arctan x$ is more rapidly convergent when x is less than 1, these other series are more useful than (2) to get good numerical approximations to π. The last such calculation before the use of computers was by Shanks in 1873. He claimed to have found π to 707 places. In the 1940s, the first computer estimate for π revealed that Shanks made a mistake in the 528-th digit, so all his further calculations were in error!

Our interest here is not to ponder ever more elaborate methods of estimating π, but to prove something about the *structure* of this number: it is irrational. That is, the number π is not the ratio of two integers. The basic idea is to argue by contradiction. We will show that if π is rational, we run into a logical error. This is also the principle behind the (much simpler) proof that the number $\sqrt{2}$ is irrational. However, there is an essential difference between the proof that $\sqrt{2}$ is irrational and the proof that π is irrational. One can prove $\sqrt{2}$ is irrational using some simple algebraic manipulations with a hypothetical rational expression for $\sqrt{2}$ to reach a contradiction. But the irrationality of π does not involve only algebra. It requires calculus. Calculus can be used to prove irrationality of other numbers, such as e and rational powers of e (excluding of course $e^0 = 1$).

The remaining sections are organized as follows. In Section 1, we prove π is irrational using some calculations with definite integrals. The irrationality of e is proved using infinite series in Section 2. A general discussion about irrationality proofs is in Section 3, and we apply those ideas to prove the irrationality of non-zero rational powers of e in Section 4.

Comments about the sample introduction

2.4 Sample Introduction (growth of functions)

Gaining an intuitive feel for the relative growth of functions is important in understanding their behavior. It also helps us better grasp topics in calculus such as convergence of improper integrals and infinite series.

In this paper, we wish to compare the growth of three different kinds of functions of x, as $x \to \infty$:

- power functions x^r for $r > 0$ (such as x^3 or $\sqrt{x} = x^{1/2}$),
- exponential functions a^x for $a > 1$,
- logarithmic functions $\log_b x$ for $b > 1$.

Some examples are plotted in Figure 1 over the interval $[1, 10]$. The relative sizes are quite different for x near 1 and for larger x. (Some coefficients are included on 2^x and x^2 to keep them from blowing up too quickly in the picture.)

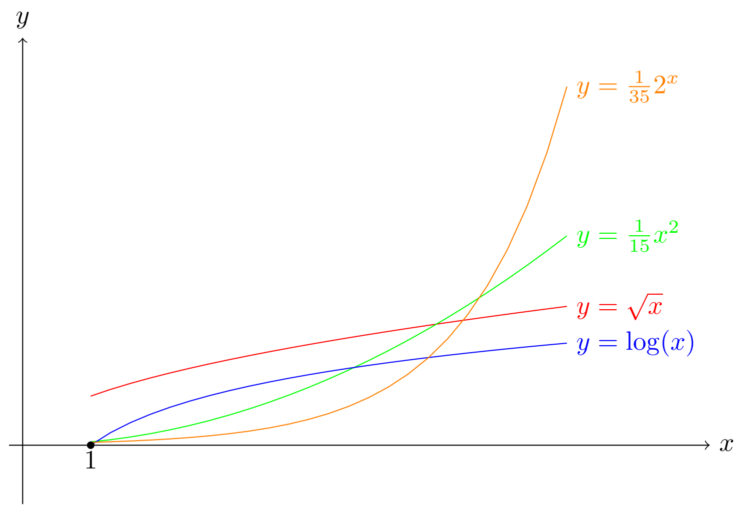

Figure 1: Graphs of several functions for $x \in [1, 10]$.

All power functions, exponential functions, and logarithmic functions (as defined above) tend to ∞ as $x \to \infty$. But these three classes of functions tend to ∞ at *different* rates. The main result we want to focus on is the following one. It says e^x grows faster than *any* power function while $\log x$ grows slower than *any* power function. (A power function means x^r with $r > 0$, so $1/x^2 = x^{-2}$ doesn't count.)

Theorem 2.4. *For each* $r > 0$, $\lim_{x\to\infty} \frac{x^r}{e^x} = 0$ *and* $\lim_{x\to\infty} \frac{\log x}{x^r} = 0$.

Theorem 2.4 is illustrated in Figure 2. At first the functions are increasing, but for larger x they tend to 0.

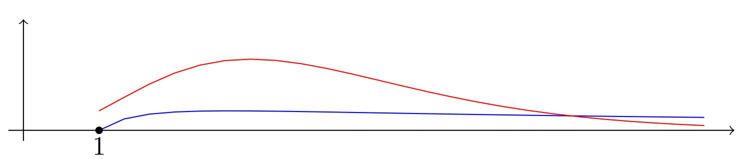

Figure 2: Graphs of x^3/e^x and $\log(x)/x$ for $x \in [1, 10]$.

In this paper, we first prove Theorem 2.4 and look at some of its consequences in Section 2. In Section 3, we will compare power, exponential, and log functions with the sequences $n!$ and n^n. Finally, we will show that between any two functions with different orders of growth we can insert infinitely many functions with different orders of growth between them.

Comments about the sample introduction

2.5 Sample Introduction (divergence)

One of the proofs of the divergence of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$$

is based on comparing the partial sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \tag{3}$$

to the integral $\int_1^n dt/t$. That they might be comparable is suggested by the integral test, which says an infinite series $\sum_{k=1}^{\infty} f(k)$ converges or diverges (under reasonable conditions) in the same way as the integral $\int_1^{\infty} f(t)\,dt$. Thus we can expect the sum (3) is approximately as large as $\int_1^n dt/t = \log n$, which diverges as $n \to \infty$. Euler discovered in 1740 that in fact the difference between (3) and $\log n$, namely

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n, \tag{4}$$

doesn't go to 0 but does have a limit as $n \to \infty$. The limit is called Euler's constant, is denoted γ, and

$$\gamma \approx .577215664901532860\ldots.$$

(Euler miscomputed the 16th digit as 5 instead of 8. See [HW96, Fig. 10.2].) So for large n, $1 + 1/2 + \cdots + 1/n \approx \log n + \gamma$.

The rest of the paper is organized as follows. In Section 2, we will show the limit of (4) exists. In Section 3, we give an error estimate so that we can prove rigorously that the decimal expansion of γ starts out as .57.

Comments about the sample introduction

3 Write your introduction

Add a draft of the introduction section to the beginning of the (revised) short paper you wrote for Bibliography Assignment 3. See the course website for instruction.

References

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[HW96] E. Hairer and G. Wanner. *Analysis by its history*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1996. Readings in Mathematics.

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