# Hilbert's 8<sup>th</sup> Problem

## Brandon Alberts

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•  $4 = 2 \cdot 2, 6 = 2 \cdot 3, \dots 236 = 2 \cdot 2 \cdot 59, \dots$ 

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1 has some very different properties compared to other positive integers, and is sometimes called a **unit**.

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Let  $N = p_1 p_2 \cdots p_n + 1$ . N > 1, so it can be written as a nonempty product of prime numbers. In particular, there exists at least one prime number  $\ell$  dividing N.

None of  $p_1, p_2, ..., p_n$  divide *N*. Thus  $\ell \neq p_1, p_2, ..., p_n$ , so  $\ell$  must be a new prime number not on our original list.

### Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
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Hilbert's  $8^{th}$  Problem

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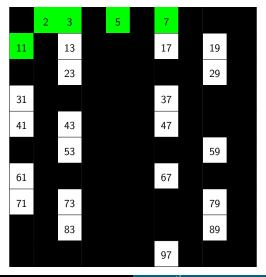
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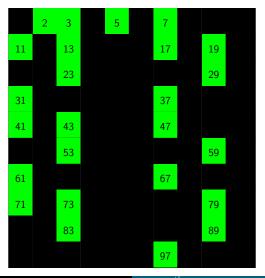
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# Primes in Arithmetic Progressions

## Theorem (Dirichlet, 1837)

Given integers a and d with share no common divisors larger than 1, the arithmetic progression

$$a, a + d, a + 2d, a + 3d, a + 4d, a + 5d, \dots$$

has infinitely many prime numbers.

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• The numbers 3 and 10 share no common divisors larger than 1: 3, 13, 23, 33, 43, 53, 63, 73, 83, 93, 103,...

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has infinitely many prime numbers.

- The numbers 3 and 10 share no common divisors larger than 1: 3, 13, 23, 33, 43, 53, 63, 73, 83, 93, 103,...
- The numbers 1 and 4 share no common divisors larger than 1: 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41,...

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We can ask a similar question about other **constellations**, such as triples (p, p + 2, p + 6) such that p, p + 2, and p + 6 are all prime.

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Trivial cases:  $(p + a_0, p + a_1, ..., p + a_k)$  such that there exists an integer n for which the remainders of  $a_0, a_1, ..., a_k$  divided by n cover all the integers 0, 1, 2, ..., n - 1 (i.e.,  $\{a_0, a_1, ..., a_k \mod n\} = \mathbb{Z}/n\mathbb{Z}$ ).

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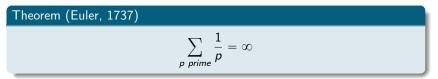
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Nontrivial cases: there is not single case which is proven!

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p

Can we show that

$$\sum_{\substack{(p,p+2)\\p,p+2 \text{ prime}}} \frac{1}{p} + \frac{1}{p+2} = \infty$$

as a way to prove the twin prime conjecture?

D

## Infinite series

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#### Theorem (Brun, 1919)

$$\sum_{\substack{(p,p+2)\\p,p+2 \text{ prime}}} \frac{1}{p} + \frac{1}{p+2} < \infty$$

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In particular,  $\frac{p_{n+1}-p_n}{\log(p_n)}$  is an unbounded sequence.

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- A polymath project has improved this number to 246, refining Zhang's approach.



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(Go to wolframcloud.com)

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This says that the percentage of primes below x is about  $1/\ln(x)$ .

We can interpret this "heuristically" to say that the probability that *n* is prime is  $1/\ln(n)$ .



• The probability that *n* is prime is heuristically  $1/\ln(n)$ .

- The probability that *n* is prime is heuristically  $1/\ln(n)$ .
- The probability of both n and n + 2 being prime is heuristically

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• The expected number of pairs (n, n + 2) for which both n and n + 2 are prime is

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• This is a divergent sum by comparison with 1/n.



# A "better" Prime Number Theorem



Prime Gaps

Prime Number Theorem

Goldbach's Conjecture

# A "better" Prime Number Theorem

Define the logarithmic integral function by

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)} \, dt \, .$$

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### A "better" Prime Number Theorem

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$$\lim_{x \to \infty} \frac{\pi(x)}{\operatorname{Li}(x)} = 1$$

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What makes this better?

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What makes this better?

$$|\pi(x) - \operatorname{Li}(x)|$$
 is "small"

14/20

Prime Gaps

Prime Number Theorem

Goldbach's Conjecture

# A "better" Prime Number Theorem

Define the logarithmic integral function by

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)} dt \, .$$

#### Theorem (Prime Number Theorem)

$$\lim_{x \to \infty} \frac{\pi(x)}{\operatorname{Li}(x)} = 1$$

Also written:

 $\pi(x) \sim \operatorname{Li}(x) \, .$ 

What makes this better?

$$|\pi(x) - \operatorname{Li}(x)|$$
 is "small"

(Go back to wolframcloud.com)



Conjecture (Riemann Hypothesis)

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- What is  $\zeta(s)$  at other complex numbers?
- What does this have to do with the prime numbers?



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- s = 1, because  $\zeta(s)$  has a singularity at s = 1,
- $s = \rho$  a zero of  $\zeta(s)$ , because  $\log(z)$  has a singularity at z = 0.

arithmetic

### complex analysis

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arithmetic	complex analysis
prime numbers	$P(s) = \sum_{p \text{ prime}} \frac{1}{p^s}$



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#### Theorem

Let a be the largest real part of a zero of  $\zeta(s)$ . Then for every  $\epsilon > 0$  there exists a constant C such that

$$|\pi(x) - \operatorname{Li}(x)| \leq C \cdot x^{a+\epsilon}$$

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# Goldbach's Conjecture

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#### Conjecture (weak Goldbach)

Every odd integer > 5 can be written as the sum of three primes.

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• The weak Goldbach conjecture is true (Helfgott, 2013)

# Thanks for coming!!

