Hilbert's 14th

- $x \leftrightarrow y \rightsquigarrow p(x,y) \xrightarrow{\sigma} p(y,x)$ transformation of $\mathbb{R}[x,y]$.

$\mathbb{R}[x,y]^\sigma := \{ p(x,y) \mid p = \sigma p \}$
$\qquad\qquad\qquad\qquad p(x,y) = p(y,x)$

<u>Claim</u>: $\mathbb{R}[x,y]^\sigma = \mathbb{R}[\underbrace{x+y}_{s}, \underbrace{xy}_{p}]$

Clearly $p(x+y, xy)$ is symmetric in x, y

$: (x+y)^4 + (x+y)xy = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 + x^2y + xy^2$

Converse also holds:

e.g. $x^5 + y^5 + x^3y + xy^3 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) + xy(x^2+y^2)$

$= (x+y)((x^2+y^2)^2 - xy(x^2+y^2+xy))$
$= (x+y)(((x+y)^2 - 2xy)^2 - xy((x+y)^2 - xy))$
$= s((s^2-2p)^2 - p(s^2-p))$

- $x \longrightarrow x+y \qquad p(x,y) \xrightarrow{\sigma} p(x+y, y)$
  $y \longrightarrow y$

$\mathbb{R}[x,y]^\sigma = \mathbb{R}[y]$

$p(x+ny, y) = p(x,y) \quad \forall n \Rightarrow$ Same for every homogeneous component

WLOG p homogeneous degree m

$\frac{p(x+ny,y)}{y^m} = \frac{p(x,y)}{y^m} \Rightarrow q(\underset{\overset{\shortparallel}{\frac{x}{y}}}{z} + n) = q(z) \quad \forall n$

z_0 complex root $\Rightarrow z_0 + n$ root $\forall n \Rightarrow q$ has ∞ly many roots

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$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc \neq 0 \qquad A$ invertible

$A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$x \longrightarrow ax + by$
$y \longrightarrow cx + dy$ $\rightsquigarrow p(x,y) \xmapsto{\sigma} p(ax+by, cx+dy)$ reversible transformation of $\mathbb{R}[x,y]$.

Hilbert 14 for $\mathbb{R}$: Let $G \subset GL_n(\mathbb{R})$ subset

$\underbrace{\mathbb{R}[x_1,\dots,x_n]}_{R}{}^G := \{ p(x_1,\dots,x_n) \mid \sigma p = p \ \ \forall \sigma \in G \}$ ← invariant polynomials

Does there exist some $p_1,\dots,p_m \in \mathbb{R}[x_1,\dots,x_n]^G$ such that every G-invariant p can be expressed polynomially in terms of $p_1,\dots,p_m$?

$\Leftrightarrow$ $\mathbb{R}[x_1,\dots,x_n]^G$ always finitely generated as $\mathbb{R}$-algebra?

Reality check: Not all subsets of $\mathbb{R}[x_1,\dots,x_n]$ that are closed under products, sums, and scaling by real constants are finitely generated

Ex: $\mathbb{R}[xy, xy^2, xy^3, \dots] \subseteq \mathbb{R}[x,y]$

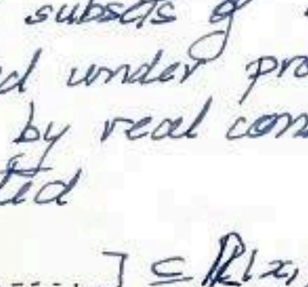
Positive examples

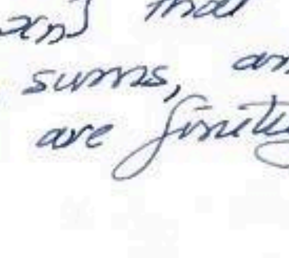
1) S_n = permutations of $x_1,\dots,x_n$

$$R^G = \mathbb{R}[\underbrace{x_1+\dots+x_n, \sum_{i<j} x_ix_j, \sum_{i<j<k} x_ix_jx_k, \dots, x_1\cdots x_n}_{\text{elementary symmetric polynomials}}]$$

2) D_n = symmetries of regular n-gon

5-gon





5 rotations, 5 reflections

4 rotations, 4 reflections

$$R_4^{D_4} = \mathbb{R}[x_1+x_2+x_3+x_4,\ x_1x_3+x_2x_4,\ x_1x_2+x_2x_3+x_3x_4+x_4x_1,\ x_1^2+x_2^2+x_3^2+x_4^2]$$

3) Chevalley: For finite groups acting true on $\mathbb{R}^n$ by reflections, get a polynomial ring. Grading is different from usual polynomial ring.

4) (Noether) G finite group, then R^G finitely generated
Proof is not constructive.

5) $GL_n = G$ $\quad R^G = R$

6) $SL_n = G$ $\quad R^G = \quad p[x,u$
$n \geq 2$

SL_2 on $k[x,y]$ $\qquad p(x,y) = p(\lambda x, \frac{1}{\lambda} y) \quad \forall \lambda \in \mathbb{R}^*$

$\cup$

$\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{N} \} \in \begin{matrix} x \to x + ny \\ y \to y \end{matrix}$

SL_n has interesting action on R_{n^2}, $\quad R_{n^2}^{SL_n} = \mathbb{R}[\det]$

6) $\mathbb{C}^* = SL_1$ $\quad R_n^{\mathbb{C}^*} = R$

7) (Linearly Reductive groups) can break apart $(R_n)_m$ = homogeneous deg m polys in $k[x_1, \ldots x_n]$ into invariant subspaces that cannot be further broken apart.

Ex: GL_n, SL_n, $(\mathbb{R}^*)^n$, finite groups

Ex $\quad \mathbb{R} \approx \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \}$ acts on $\mathbb{R}^2$

The only proper invariant subspace: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$
is $\{ \begin{pmatrix} u \\ v \end{pmatrix} \} = \{ \begin{pmatrix} u \\ 0 \end{pmatrix} \mid u \in \mathbb{R} \}$

So $(\mathbb{R}, +)$ is not reductive.

Counterexamples exist

- Nagata $(\mathbb{R}^{13}, +)$
- Mukai $(\mathbb{C}^3, +)$
- Totaro $(\mathbb{R}^3, +) \circlearrowright \mathbb{R}^{18}$

$\mathbb{R}^4 \circlearrowright \mathbb{R}^{16}$

$\mathbb{R}^6 \circlearrowright \mathbb{R}^{18}$

Counter examples of Totaro (Mukai) have the following flavor

$n \geq r \geq 3$

$G = (\mathbb{R}^{n-r}, 0) \subseteq \mathbb{R}^n$

$\mathbb{R}^n$ acts on $\mathbb{R}^{2n}$ by $(t_1, \ldots, t_n)(x_1, \ldots, x_n, y_1, \ldots, y_n)$

$\rightarrow (x_1, \ldots, x_n, y_1 + t_1 x_1, \ldots, y_n + t_n x_n)$

$\mathbb{R}^n$ are transformations of $k[x_1, \ldots, x_n, y_1, \ldots, y_n] = R_{2n}$

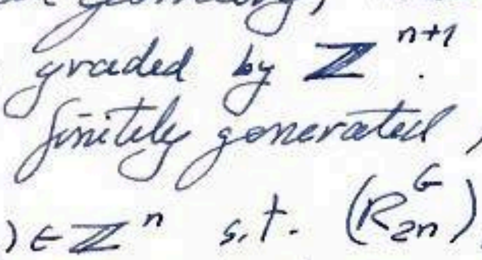
Using algebraic geometry, Mukai showed

R_{2n}^G is graded by $\mathbb{Z}^{n+1}$.

If R_{2n}^G is finitely generated, then

$(a_0, \ldots, a_n) \in \mathbb{Z}^n$ s.t. $(R_{2n}^G)_{(a_1, \ldots, a_n)} \neq 0$ would

generate a polyhedral cone in $\mathbb{R}^{n+1}$



← finitely many "corners", flat otherwise.

But when $\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \leq 1$, can get infinitely many corners or maybe roundness.