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HILBERT'S FIRST PROBLEM: THE CONTINUUM HYPOTHESIS

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ABSTRACT

The present status of the continuum hypothesis and the generalized continuum hypothesis is discussed. Both independence results and recent positive theorems are listed. An analysis is given of the bearing on the continuum problem of work on large cardinals and projective sets.

1. INTRODUCTION

Hilbert's First Problem is in the curious position that there is serious disagreement as to whether it has been solved and there is related disagreement as to whether the problem, in the natural way of understanding it, is a mathematical problem at all.

The First Problem is to settle the famous continuum hypothesis (CH) of Cantor. CH asserts that the cardinal number of the continuum (the set of all real numbers) is $\aleph_1$, the smallest uncountable cardinal number. Equivalently, CH states that there are the same number of real numbers as countable ordinal numbers.

The generalized continuum hypothesis (GCH) asserts that, for every infinite cardinal number $\aleph_\alpha$, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. In other words, the cardinal number of the collection of all subsets of a set of cardinality $\aleph_\alpha$ is the smallest cardinal number greater than $\aleph_\alpha$. CH is the special case $2^{\aleph_0} = \aleph_1$.

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Hilbert [13] himself attempted to prove CH, but he was unable to carry the proof through to completion. In 1938 Gödel [9,10] attacked the problem in a very surprising manner. He showed that, if the standard Zermelo-Fraenkel (ZFC) axioms for set theory are consistent, then there can be no refutation of GCH from these axioms. (There is a relation -- however tenuous -- between Gödel's proof and Hilbert's unsuccessful attempt to prove CH.) Gödel [11] conjectured that the formal ZFC axioms do not suffice to prove CH either and thus that CH is formally undecidable in the theory ZFC. In 1963 Cohen [4] proved that this is the case.

Where do these results leave Hilbert's First Problem? From Hilbert's formalist standpoint CH is an assertion of ideal mathematics rather than of real mathematics. Hilbert, however, presumably thought that the formal axioms of set theory were strong enough to settle such propositions as CH. It is unclear whether he would regard the Gödel and Cohen results as a solution to his problem. Gödel [11] holds that the meaning of CH is independent of formal axioms and that independence proofs only show the weakness of our current axioms. In his view, CH is either true or false, and the problem of discovering which is still with us. Cohen [5] espouses a formalist position but still holds that we may yet be led to a decision about CH, specifically that we may be led to accept the negation of CH as an axiom.

2. CH AND GCH IN ZFC

I shall return to these essentially philosophical questions later, but first I wish to discuss the mathematical work related to CH and GCH which grew out of the independence proofs.

Gödel's work, for reasons which are unclear, was followed by twenty years of stagnation in the field of set theory. Though his proof had introduced new concepts and techniques in set theory and an interesting new proposition, the axiom of constructibility, little use was put to these new ideas until after the work of Cohen. Instead his theorem had a negative effect on some branches of set theory. For example, Gödel showed that certain basic propositions considered in classical descriptive set theory could not be proved in ZFC. Hence there was some danger that many of the important questions in this area were formally undecidable. This tended as a practical matter to discourage work in the field, and few basic advances were made (excepting Addison [1,2] and work of Choquet) until recently.

Cohen's proof had quite the opposite effect. It ushered in a period of intense activity in set theory. Cohen's methods were applied to every imaginable set theoretic question, and a great number of questions were shown to be undecidable in ZFC. There was even a revival of interest in Gödel's axiom of constructibility, and many important consequences of this axiom were deduced (mostly by Ronald Jensen).

The effect of independence proofs on mathematics is not entirely negative. For example, there are several cases of theorems having been proved assuming CH where independence proofs allow one to eliminate the hypothesis CH. For (a class of) example(s), given any proposition ϕ of second order number theory (loosely speaking: any proposition about integers and reals only -- e.g., not about arbitrary sets of reals) if ϕ is provable from ZFC + CH then ϕ is provable from ZFC (Platek [24], S. Kripke, J. Silver). A similar theorem is true about the negation of CH, though one must restrict ϕ to be an assertion about integers alone. Such theorems are special cases of a general absoluteness technique. Suppose one can prove in ZFC that a proposition ψ implies another proposition ϕ. It is sometimes the case that, given any model of ZFC, one can extend it by Cohen's methods to a model of ZFC + ψ. In the larger model ϕ must be true. But if ϕ can be proved to be sufficiently absolute, one can conclude that ϕ must be true in the original model and hence that ZFC implies ϕ. Absoluteness of ϕ means that the truth value of ϕ is preserved under all or a wide class of such Cohen extensions of models.

Before the work of Cohen and Gödel, one important restriction on the operation $\aleph_\alpha \to 2^{\aleph_\alpha}$ was known. This result of König [15] states that $2^{\aleph_\alpha}$ cannot have cofinality $\leq \aleph_\alpha$. The cofinality of a cardinal κ is the least cardinal λ such that a set of cardinality κ is always the union of λ sets of cardinality smaller than κ. In particular $2^{\aleph_0}$ cannot equal $\aleph_\omega$, where ω is the first infinite ordinal number.

Cohen and R. Solovay showed (using models constructed by Cohen) that König's restriction is the only restriction on $2^{\aleph_0}$. Easton [7] attacked the GCH using Cohen's methods. He almost showed that the operation $\aleph_\alpha \to 2^{\aleph_\alpha}$ can be anything consistent with König's theorem. The "almost" basically involves the problem of singular cardinals. κ is singular if the cofinality of κ is smaller than κ. Easton could produce models of ZFC with $\aleph_\alpha \to 2^{\aleph_\alpha}$ whatever he wished on the class of regular (non-singular) $\aleph_\alpha$ but could not at the same time control the values at singular $\aleph_\alpha$. For example, it is not known whether it is consistent with ZFC that $2^{\aleph_n} = \aleph_{n+1}$ for $n < \omega$ and $2^{\aleph_\omega} > \aleph_{\omega+1}$. (One restriction in addition to König's has been known for some time: If κ is singular and $2^\gamma = \lambda$ for all sufficiently large $\gamma < \kappa$, then $2^\kappa = \lambda$.)

One of the major problems in post-Cohen set theory has been this <u>singular cardinals problem</u>. Most workers have felt that the theorems of Easton could be extended to singular cardinals. Work of Prikry, Silver, and Magidor has led to some consistency results, but they are partial and the "consistency" is relative to theories much stronger than ZFC. Very recently Jack Silver astonished the set-theoretic world by essentially settling the singular cardinals problem for cardinals of cofinality greater than $\aleph_0$ -- and settling it in the "wrong" direction. A consequence of Silver's theorem (a theorem of ZFC alone!) is that if $\aleph_\alpha$ is singular of cofinality $> \aleph_0$ and if $2^{\aleph_\beta} = \aleph_{\beta+1}$, for all $\beta < \alpha$ then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Galvin and Hajnal [8] have extended Silver's work to compute, in a sense, absolute bounds on $2^{\aleph_\alpha}$ for certain singular $\aleph_\alpha$. The problem of singular cardinals of cofinality $\aleph_0$ remains, despite this breakthrough, as puzzling as ever.

3. LARGE CARDINAL AXIOMS

Although the ZFC axioms are insufficient to settle CH, there is nothing sacred about these axioms, and one might hope to find further axioms which seem clearly true of our notion of set (in the same way the ZFC axioms appear clearly true) and which do settle CH.

In the time since Cohen, there has been a great deal of research on one class of candidates for new axioms: the so-called <u>large cardinal</u>

axioms. (This area by no means began after Cohen, however. The subject is much older, and its revival occurred before Cohen's work. For example, [14] and [25] are pre-Cohen.) A large cardinal axiom is, roughly speaking, an assertion that cardinal numbers exist having some property P, such that one can prove that only very large cardinals can have P. Examples of such P are inaccessibility and measurability. κ is inaccessible if κ is regular and $\lambda < \kappa$ implies $2^{\lambda} < \kappa$. κ is measurable if there is a set A of cardinality κ and a function μ defined on all subsets of A such that μ takes only 0 and 1 as values, $\mu(A) = 1$, μ is 0 on singletons, $\mu(A-X) = 1 - \mu(X)$, and if $\mu(A_i) = 0$ for each $i \in I$ and cardinal $(I) < \kappa$ then $\mu(\bigcup A_i) = 0$.

The usual large cardinal axioms cannot be proved in ZFC if it is consistent. There are basically three sorts of arguments for accepting them: analogy with $\aleph_0$, reflection principles, and plausibility of consequences.

The argument from analogy with $\aleph_0$ goes as follows: $\aleph_0$ is inaccessible, measurable, etc. For each of these properties P it would be only by accident, as it were, that $\aleph_0$ should be definable as the unique infinite cardinal such that P (as it is an accident that man = featherless biped). Hence one would expect that larger cardinals having property P exist.

The argument from reflection principles starts with the usual notion of sets as generated by iteration of the power set operation. We start with R_0 = the empty set. Given R_α, for an ordinal number α, $R_{\alpha+1}$ is the collection of all subsets of R_α. For limit ordinals λ, $R_\lambda = \bigcup_{\beta<\lambda} R_\beta$. A set is anything which is a member of some R_α. The formal axioms of set theory are an attempt to describe this iterative construction. One thing we want the axioms to assert is that the construction does not stop too soon because the ordinal numbers are exhausted prematurely. The axiom of replacement is intended to assert that the ordinals do not run out at an unnatural point. Reflection principles are based on the idea that the class On of ordinal numbers is so large that, for any reasonable property P of the universe of all sets R_{On}, On is not the first stage α such that R_α has P. Examples of "reasonable" properties are first, second, and higher order properties. If this is right, then there should be stages R_α which look very much like R_{On}. It follows that there should be stages R_α and R_β which look very much alike. All important large cardinal axioms which have been studied are derivable from assertions that R_α and R_β exist which are difficult to distinguish. Of course, as the axioms become stronger their link with the basic principle becomes more and more tenuous.

A third argument for large cardinal axioms is that the theory their adoption gives is plausible and appealing. This is almost an empirical argument. I shall say more about this view later when I discuss another kind of axiom to which it better applies.

The reasons advanced for adopting large cardinal axioms are -- as the reader has surely noticed -- less compelling than the reasons for adopting ZFC. On the other hand, they are not completely negligible, and one should bear in mind that the axioms of ZFC (and the notion of set which they supposedly describe) are less compelling than the axioms of number theory.

What do large cardinal axioms tell us about the continuum hypothesis? Unfortunately they tell us very little. Large cardinal axioms tend to be absolute in the sense discussed earlier. If they are true in a model of ZFC they tend to be true in Cohen extensions of that model. This assertion can be made more precise as follows. A Cohen extension of a model M arises from an element P of M which is a partial ordering in M. A Cohen extension is <u>mild</u> with respect to a cardinal κ of M if cardinal (P) $< \kappa$ is true in M. All the standard large cardinal properties of κ are preserved under Cohen extensions mild with respect to κ. The truth value of CH, on the other hand can be changed by Cohen extensions where P has very small cardinal in M. This means that, for a large cardinal axiom A, there are models of ZFC + A + CH and models of ZFC + A + not CH, if there are models of ZFC + A at all.

Large cardinal axioms have, nonetheless, been successful in giving partial results about the GCH. Solovay [27] has shown that the existence of a so-called compact cardinal implies that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all sufficiently large singular strong limit cardinals (i.e., singular cardinals κ such that $\lambda < \kappa$ implies $2^\lambda < \kappa$) $\aleph_\alpha$.

4. PROJECTIVE SETS

We have at present no likely candidate for a new axiom which would settle CH. Let us despite this adopt for the moment the point of view that CH is a meaningful proposition and ask whether there is any information available which counts as evidence for or against the truth of CH.

Gödel [11] cites some facts which he believes are evidence against CH. He lists a number of known consequences of CH which he thinks are intuitively implausible. These consequences assert that very thin subsets of the real line exist of cardinality the continuum. Gödel says that such assertions are counterintuitive in a sense different from that in which the existence of Peano curves is counterintuitive. While Gödel's intuitions should never be taken lightly, it is very hard to see that the situation is different from that of Peano curves, and it is even hard for some of us to see why the examples Gödel cites are implausible at all.

Another way to look for evidence concerning CH is to examine simple cases. CH says that every set of reals is countable or has cardinality $2^{\aleph_0}$. We might test that assertion by looking at sets of reals which are, in some sense simple. If such simple sets do not provide counterexamples, we can try to see whether there are reasons to suspect that the simple sets considered are in a relevant way different from arbitrary sets of reals.

The simple sets I wish to consider are the projective sets. A set of reals is projective if it can be gotten from a Borel set via the operations of continuous image and complementation. The projective sets can be divided into a hierarchy as follows: Σ^1_1 (or analytic) sets are continuous images of Borel sets. Π^1_n sets are complements of Σ^1_n sets; Σ^1_{n+1} sets are continuous images of Π^1_n sets.

Now every Borel set is countable or has cardinal $2^{\aleph_0}$. The same is true of every Σ^1_1 set. Σ^1_2 sets are always unions of $\aleph_1$ Borel sets, so Σ^1_2 sets are countable or have cardinal $\aleph_1$ or $2^{\aleph_0}$. So far CH is confirmed. It is known to be consistent with ZFC that $2^{\aleph_0} > \aleph_1$ and that there are Σ^1_2 (even Π^1_1) sets of power $\aleph_1$. It is also consistent [12] that $2^{\aleph_0}$ be as large as you wish and that there are Π^1_2 sets of every cardinality $\leq 2^{\aleph_0}$. Thus ZFC furnishes us with no information about higher levels of the projective hierarchy.

Let us see if large cardinal axioms help. Let MC be the assertion that measurable cardinals $\aleph_0$ exist. Solovay has shown that MC implies that every infinite Σ^1_2 set has cardinal $\aleph_0$ or $2^{\aleph_0}$ [26]. Concerning Σ^1_3 sets, MC implies [18] that every Σ^1_3 set has cardinal $\aleph_0$, $\aleph_1$, $\aleph_2$ or $2^{\aleph_0}$.

Can we regard these facts about Σ^1_1 and Σ^1_2 sets as evidence for CH? I do not think we can. For the results about the cardinalities of Σ^1_i sets, i = 1, 2, are corollaries to stronger results. Every Σ^1_i set, i = 1, 2, is countable or has a perfect subset (assuming MC for i = 2). Now, by a simple application of the axiom of choice, there exists an uncountable set with no perfect subset. Thus, while our simple sets have the cardinalities required by CH, this is so because they have an <u>atypical</u> property, the perfect subset property.

We might try different formulations of CH. For example, CH says that every well-ordering of a subset of the reals has order type $< \omega_2$, the second uncountable initial ordinal. The notions of projective and Σ^1_n relations can be defined in the obvious way, and we can ask about the order type of Σ^1_n well-orderings. The relevant theorems are that Σ^1_i well-orderings are countable for i = 1 or 2, assuming MC for i = 2. In other words, the evidence suggests that simple well-orderings are countable and cannot even have order type ω_1. Once again our simple sets have proved atypical.

There is a third formulation of CH which is more promising. The negation of CH says that there is a <u>surjection</u>

$$f : R \to \omega_2 ;$$

where R is the reals and ω_2 is thought of as the set of its predecessors. Now a function

$$f : R \to \text{Ordinals}$$

is essentially the same as a prewellordering of R. To prewellorder a set, divide it into equivalence classes and well-order the equivalence classes. By the length of a prewellordering, I mean the order-type of the well-ordering of the equivalence classes. Now every Σ_1^1 prewellordering has countable length but there is a Π_1^1 prewellordering of R of length ω_1 (essentially the Lebesgue decomposition of R). This already shows that our simple sets are more typical with respect to prewellorderings than with respect to well-orderings.

Let δ_n^1 be the least ordinal > 0 not the length of a Σ_n^1 prewellordering of R. Then we have

$$\delta_1^1 = \omega_1 \quad ;$$

$$\delta_2^1 \leq \omega_2 \quad ;$$

$$MC \rightarrow \delta_3^1 \leq \omega_3 \quad .$$

(See [18] for the last two results.) Neither $\leq$ can be improved (in ZFC + MC) to $=$ since CH implies that $\delta_n^1 < \omega_2$ for all n. On the other hand $\delta_2^1 = \omega_2$ is consistent with ZFC. This follows from a result in [19]. Thus, while our simple sets have not provably given us a counterexample to CH, the possibility that they are counterexamples definitely arises.

Related theorems give a similar picture:

$$\text{Every } \Sigma_2^1 \text{ set is a union of } \aleph_1 \text{ Borel sets.}$$
$$MC \rightarrow \text{Every } \Sigma_3^1 \text{ set is a union of } \aleph_2 \text{ Borel sets.}$$

Once again it is consistent that these results are not best possible, but there is no reason to believe they are not best possible.

Measurable cardinals do not give information about higher levels of the projective hierarchy, but there is another sort of "axiom" which does. This is the assertion projective determinacy. Let 2^ω be the collection of all infinite sequences of 0's and 1's. Regard 2^ω as a product of ω copies of the discrete space $\{0,1\}$ and give it the product topology. Given $A \subseteq 2^\omega$, the game G_A is defined as follows. Players I and II take turns picking 0 or 1, thus producing an element of 2^ω. I wins if this element belongs to A. The notion of a winning strategy for I or II is defined in the obvious way. G_A is determined if one of the players has a winning strategy. Using the axiom of choice, one can construct an A such that G_A is not determined (See [21]). On the other hand we have recently proved that G_A is determined for every Borel set A (the best previous result [23] concerned $F_{\sigma\delta\sigma}$ sets), and MC implies [17] that G_A is determined for every Π^1_1 set A. (The projective hierarchy is defined in the same way as for the reals.) _Projective determinacy_ (PD) is the assertion that G_A is determined for every projective A.

There is no _a priori_ evidence for PD, but there is a good deal of _a posteriori_ evidence for it. PD has pleasing consequences about the behavior of projective sets, such as: Every projective set is Lebesgue measurable [22]; Every uncountable projective set has a perfect subset [6]. More impressive is the fact that PD allows one to extend the classical structural theory of projective sets, which dealt only with the first two levels of the projective hierarchy, to a very elegant and essentially complete theory of the projective sets (See [3], [16], [20]). PD cannot be proved in ZFC (or ZFC + MC, though it may be provable from large cardinal axioms), but it is not unreasonable to suspect that it may be true.

All the consequences of MC concerning the projective sets also follow from PD. Furthermore [18]

$$PD \rightarrow \delta^1_4 \leq \omega_4 \quad ;$$

$$PD \rightarrow \text{Every } \Sigma^1_4 \text{ set is a union of } \aleph_3 \text{ Borel sets.}$$

Concerning higher levels one has

$$PD \rightarrow \delta^1_{2n+2} \leq (\delta^1_{2n+1})^+ \quad ,$$

where α^+ is the least initial ordinal greater than α. Thus PD extends the pattern derived from MC (which extends that derived in ZFC alone) and reinforces the suggestion that the answer to CH may be negative.

Throughout the latter part of my discussion, I have been assuming a naive and uncritical attitude toward CH. While this _is_ in fact my attitude, I by no means wish to dismiss the opposite viewpoint. Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty.

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