Magical Numbers and Where to Find Them

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Contents

Abstract

Every rational number can be categorized as one of two types: algebraic or transcendental. This paper will discuss properties of these two types as well as attempt to categorize two famous constants, e and π. We will then apply our results to answer a classic question in geometry.

1 Introduction

Abstract algebra is a field that results from stripping away artificial structures we impose on our own number systems, and building up from basic concepts. Ultimately, we reduce everything to sets with operations on them, and yet this can give us surprisingly deep and fundamental results in every area of mathematics.

This paper is organized as follows. In Section 2, we will discuss how algebra can be used to organize all numbers into two categories: algebraic and transcendental. In Section 3, we will place the famous constants e and π in the latter, and then prove the impossibility of the ancient question of squaring the circle.

2 Background

2.1 Fields

We review some basic concepts from field theory. A more comprehensive introduction to fields can be found in [BB06].

Definition 2.1. *Let F be a set equipped with addition $(+)$ and multiplication $(\cdot)$. F is a* **field** *if the following properties.*

1. $(F, +)$ *is an abelian group.*
2. $(F \setminus \{0\}, \cdot)$ *is an abelian group.*
3. $a \cdot (b + c) = a \cdot b + a \cdot c$ *for* $a, b, c \in F$.

Loosely speaking, a field is a set in which you can add, subtract, multiply, and divide any element with any other element (except dividing by 0) and remain in the set.

Definition 2.2. *Let $K \subseteq F$. Then F is called an* **extension field** *of K if K is a field under the operations of F. In this case, K is called the* **base field**.

2.2 Algebraic and Transcendental Numbers

Definition 2.3. *Let F be an extension field of K and let $u \in F$. If there exists a polynomial $f(x)$ with coefficients in K such that $f(u) = 0$, then u is said to be* **algebraic** *over K. If no such polynomial exists, then u is said to be* **transcendental** *over K.*

Example 2.4. *The constant $\sqrt{2}$ is algebraic over $\mathbb{Q}$ because it is a root of the polynomial $x^2 - 2$.*

Theorem 2.5. *Algebraic numbers (over $\mathbb{Q}$) form a field.*

Proof. Let α, β be algebraic numbers and let m, n be the minimum integers such that the following polynomials are irreducible (not factorable).

$$\alpha^m = a_{m-1}\alpha^{m-1} + \cdots + a_1\alpha + a_0$$
$$\beta^n = b_{n-1}\beta^{n-1} + \cdots + b_1\beta + b_0$$

Now consider the $mn+1$ numbers:

$$1, \alpha+\beta, (\alpha+\beta)^2, \cdots, (\alpha+\beta)^{mn}$$

With some manipulation, we see that these $mn+1$ numbers can be written as $\mathbb{Q}$-linear (having coefficients in $\mathbb{Q}$) combinations of the mn numbers $\alpha^i\beta^j$ for $i = 0, 1, \cdots, n-1$ and $j = 0, 1, \cdots, m-1$. Thus the $mn+1$ numbers above are linearly dependent and $\alpha + \beta$ is algebraic.

The proof that $\alpha\beta$ is algebraic is similar.

So we have shown that algebraic numbers are closed under addition and multiplication, and it is clear to see that they are also closed under subtraction and division by nonzero elements. □

Corollary 2.6. *The complex number $a+bi$ is algebraic over $\mathbb{Q}$ if and only if a and b are both algebraic over $\mathbb{Q}$.*

Proof. Suppose a and b are algebraic. Note that i is the root of the polynomial $x^2+1=0$, thus i is algebraic. So $a+bi$ is a linear combination of algebraic numbers which is algebraic by Theorem 2.5.

If $a+bi$ is algebraic and $f(a+bi)=0$ for some polynomial $f(x)$ over $\mathbb{Q}$, then $f(a-bi)=0$ because the coefficients must be rational. So

$$a = \frac{1}{2}((a+bi)+(a-bi)).$$
$$b = \frac{1}{2i}((a+bi)-(a-bi))$$

Thus a and b are both algebraic. □

Theorem 2.7. *Algebraic numbers are countable*

Proof. Enumerate polynomials of the form $f(x) = a_n x^n + \cdots a_1 x + a_0$, with $a_j \in \mathbb{Q}$, by

$$i_f = |a_n| + |a_{n-1}| + \cdots + |a_0|.$$

Since $\mathbb{Q}$ is countable, these polynomials are countable. By the fundamental theorem of algebra, each index i_f corresponds to finitely many (at most $\deg f$) new algebraic numbers. Furthermore, we obtain every algebraic number this way because every polynomial has an index, so algebraic numbers are countable. □

Corollary 2.8. *Almost all real numbers are transcendental.*

Proof. Since $\mathbb{R}$ is uncountable and algebraic numbers are countable by Theorem 2.7, transcendental numbers must be uncountable. □

3 Transcendence of e and π

3.1 Important Theorems

We describe a few important theorems on the properties of e and π.

Theorem 3.1. *The numbers e and π are transcendental over $\mathbb{Q}$*

A full proof of this theorem can be found in [Niv05]. The proof for e was originally published by Charles Hermoite in 1873, and the proof for π was originally proved by Ferdinand Lindemann in 1882. Ultimately, the proof that π is transcendental follows from the following theorem, proved by Lindemann.

Theorem 3.2. *Given distinct algebraic numbers $\alpha_1, \cdots, \alpha_n$, the numbers $e^{\alpha_1}, \cdots, e^{\alpha_n}$ are linearly independent.*

Then, if π were algebraic, $i\pi$ would be algebraic, and $e^{i\pi} = -1$ would be transcendental, a contradiction. Thus π is transcendental. Similarly, for any algebraic α, trigonometric and inverse trigonometric functions, as well as the logarithm of α (e.g. $\sin(\alpha), \cos^{-1}(\alpha), \log(\alpha)$) are all transcendental.

This raises an interesting question. By Theorem 2.5, sums and products of algebraic numbers are also algebraic, but the same cannot be said for transcendental numbers. It is actually an open question whether or not the values $\pi + e$ or πe are transcendental, or even irrational.

However, we can say the following:

Theorem 3.3. *At least one of $e + \pi$ or $e\pi$ is irrational.*

Proof. Supposed $e + \pi$ and $e\pi$ are rational. Then the polynomial $x^2 + (e + \pi)x + e\pi$ has rational coefficients and is satisfied by e and π, contradicting the fact that e and π are transcendental over $\mathbb{Q}$. □

3.2 Squaring the Circle

Lindemann's proof can be applied to a question posed by ancient geometers:

Question 3.4. *Is it possible to construct a square with the same area as a circle with only straightedge and compass?*

Lindemann's proof shows that the answer is indeed **no**. All geometric constructions have algebraic lengths, so constructing a line of length $\sqrt{\pi}$ would mean that $\sqrt{\pi}$ is algebraic, a contradiction.

References

[BB06] J.A. Beachy and W.D. Blair. *Abstract Algebra*. Waveland Press, 2006.

[Niv05] I. Niven. *Irrational Numbers*. Carus Mathematical Monographs. Mathematical Association of America, 2005.