Friezes, triangulations, continued fractions, and binary numbers

Emily Gunawan
University of Connecticut

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Slides available at
https://egunawan.github.io/talks/smith18

Frieze patterns

A *frieze pattern* is an image that repeats itself along one direction. The name comes from architecture, where a frieze is a decoration running horizontally below a ceiling or roof.

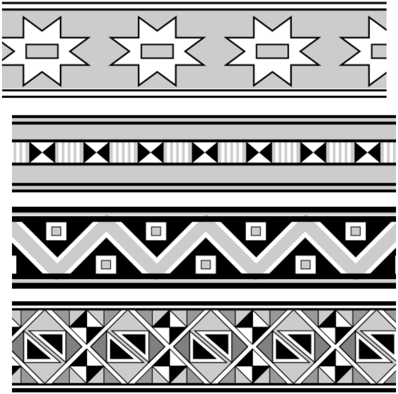


Figure: M. Ascher, *Ethnomathematics*, p162.

Conway-Coxeter frieze patterns

Definition

A (Conway-Coxeter) **frieze pattern** is an array such that:

1. the top row is a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the rule $ad - bc = 1$.

Example (an integer frieze)

$$\begin{array}{lccccccccccccccccc}
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\text{Row 2} & & \cdots & & \mathbf{3} & & \mathbf{1} & & \mathbf{2} & & \mathbf{2} & & \mathbf{1} & & 3 & & 1 & \\
 & & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
 & & \cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Note: every frieze pattern is completely determined by the 2nd row.

Glide symmetry

A *glide symmetry* is a combination of a translation and a reflection.

		1		**1**		**1**		**1**		**1**		**1**		**1**		…
Row 2	…		**3**		**1**		**2**		**2**		**1**		**3**		**1**	
		2		**2**		**1**		**3**		**1**		**2**		**2**		…
	…		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

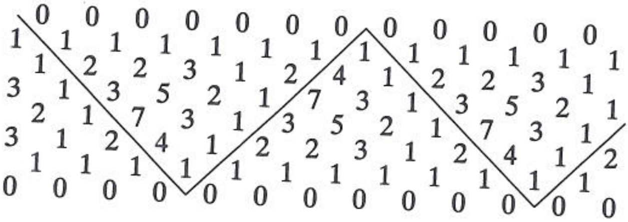


Table 1.

Practice

Frieze 1

0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1

1 2 2 3 1 2 4 1 2 2 3 1

3 5

7

Rule

Every

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

satisfies

$$ad - bc = 1$$

Frieze 2

0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

2 2 2 3 1 2 5 1 2 2 2 3 1

1 3 3 9

4

Practice: Answer Key

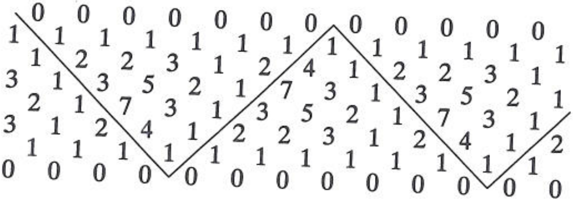


Table 1.

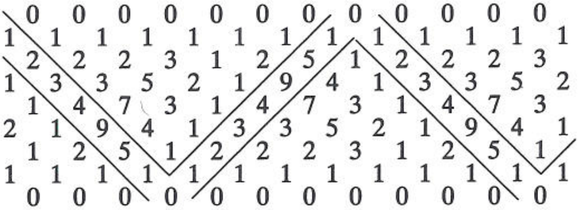


Table 2.

What do the numbers around each polygon count?

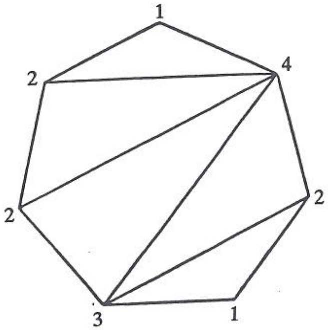
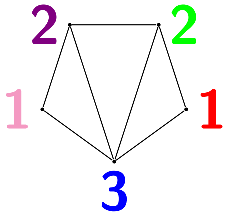


Figure 3: A triangulation of a heptagon.

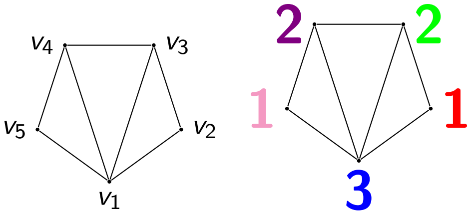


Conway and Coxeter (1970s)

Theorem

Finite frieze patterns with positive integer entries $\longleftrightarrow$ *triangulations of polygons*

$$
\begin{array}{lccccccccccccccccc}
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\text{Row 2} & \cdots & & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & \\
 & & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
 & \cdots & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$



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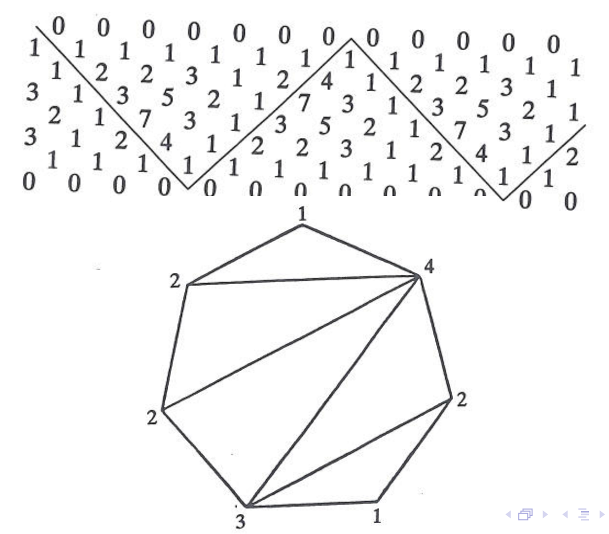
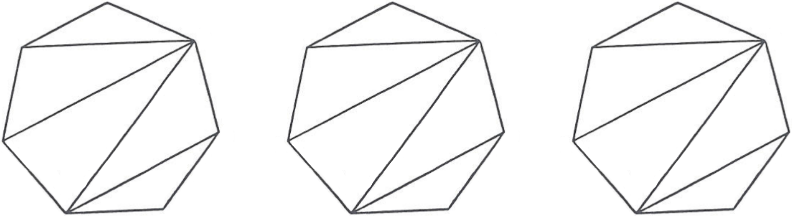


Primary school algorithm

	0		0		0		0		0		0		0		0		0		0		0		0	
1		1		1		1		1		1		1		1		1		1		1		1		1
	1		2		2		3		1		2		4		1		2		2		3		1	
3		1		3		5		2		1		7		3		1		3		5		2		1
	2		1		7		3		1		3		5		2		1		7		3		1	
3		1		2		4		1		2		2		3		1		2		4		1		2
	1		1		1		1		1		1		1		1		1		1		1		1	
0		0		0		0		0		0		0		0		0		0		0		0		0

Table 1.

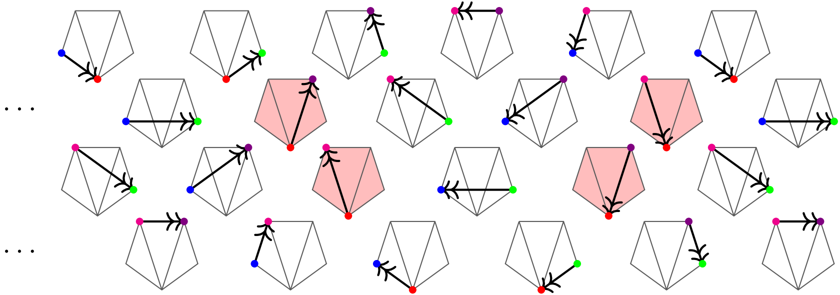


Broline, Crowe, and Isaacs (BCI, 1970s)

Theorem

Entries of a finite frieze pattern $\longleftrightarrow$ edges between two vertices.

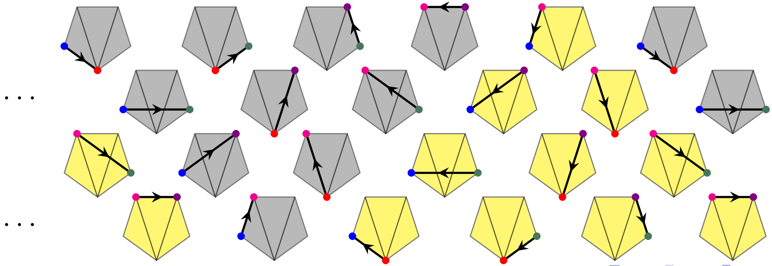
		1		1		1		1		1		1		⋯
Row 2	⋯		3		**1**		2		2		**1**		3	
		2		2		**1**		3		**1**		2		⋯
	⋯		1		1		1		1		1		1	



Glide symmetry (again)

A *glide symmetry* is a combination of a translation and a reflection. If we forget the arrows' orientation, the diagonals have glide symmetry.

		1		**1**		**1**		**1**		**1**		**1**		**1**		···
Row 2	···		**3**		**1**		**2**		**2**		**1**		**3**		**1**	
		2		**2**		**1**		**3**		**1**		**2**		**2**		···
	···		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

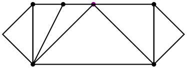


Binary numbers

A binary number is a number expressed in the base-2 numeral system, with digits consisting of 1s and $0s$. For example,

- $1 = 1 * 2^0$ (in decimal) is written as 1 (in binary).
- $2 = 1 * 2^1$ (in decimal) is written as 10 (in binary).
- $4 = 1 * 2^2$ (in decimal) is written as 100 (in binary).
- $5 = 1 * 2^2 + 1 * 2^0$ (in decimal) is written as 101 (in binary).
- $29 = 16 + 8 + 4 + 1 = 1 * 2^4 + 1 * 2^3 + 1 * 2^2 + 1 * 2^0$ is written as 11101 (in binary).

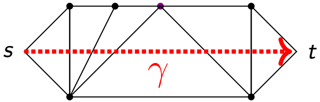
Subwords of binary numbers (G.)



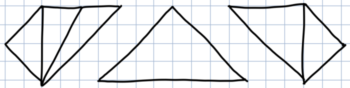
We can think of “11101” as a word in the alphabets $\{0, 1\}$. All subwords of “11101” which start with “1”:

- “11101” (itself)
- “11101”
- “11101”
- “11101”
- “11101”
- “11101”
- “11101”
- “11101”
- “11101”
- “11101”
- “11101”

Continued fractions (Çanakçı, Schiffler)



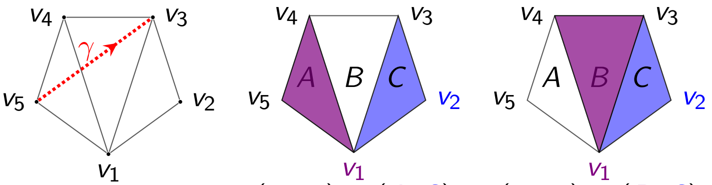
Continued Fraction



3 , 1 , 2

$$3 + \frac{1}{\left(1 + \frac{1}{2}\right)} \overset{\text{compute}}{=}$$

Broline, Crowe, and Isaacs (BCI, 1970s)



T and γ $\quad (t_1, t_2) = (A, C) \quad (t_1, t_2) = (B, C)$

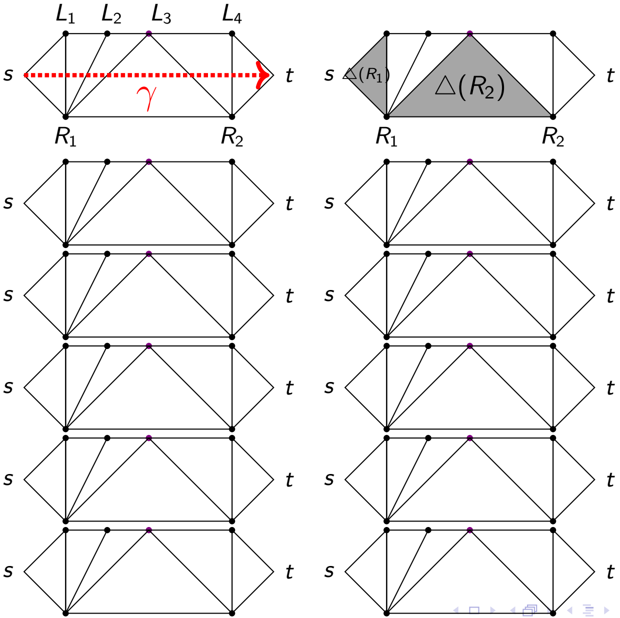
Definition (BCI tuple)

Let R_1, R_2, ..., R_r be the boundary vertices to the right of γ. A **BCI tuple** for γ is an r-tuple $(t_1, \ldots, t_r)$ such that:

(B1) the i-th entry t_i is a triangle of T having R_i as a vertex.

(B2) the entries are pairwise distinct.

How many BCI tuples are there?



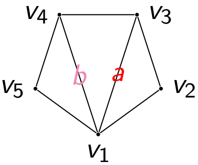
Cluster algebras (Fomin and Zelevinsky, 2000)

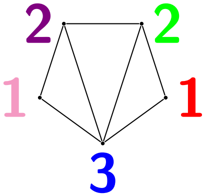
Definition A **cluster algebra** is a commutative ring with a distinguished set of generators, called **cluster variables**.

Theorem (Caldero-Chapoton (2006): The cluster variables of a cluster algebra from a triangulated polygon (type A) form a frieze pattern.

$$
\begin{array}{ccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & \\
\cdots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

(Example: type A_2)





- Remark: If the variables are specialized to 1, we recover the integer frieze pattern.

Thank you