Frieze vectors and unitary friezes

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I. Friezes

Let Q be a quiver and $\mathcal{A}(Q)$ the cluster algebra from Q.

- A positive integral ***frieze*** of type Q is a ring homomorphism $F : \mathcal{A}(Q) \to R = \mathbb{Z}$ which maps every cluster variable to a positive integer.
- A positive integral frieze is called ***unitary*** if there exists a cluster $\mathbf{x}$ in $\mathcal{A}(Q)$ such that F maps every cluster variable in $\mathbf{x}$ to an invertible element in R, i.e., $\mathcal{F}(x_i) = 1$ for each $x_i \in \mathbf{x}$ since 1 is the only unit in $\mathbb{Z}_{>0}$.

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & \\
& x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & \frac{x_2+1}{x_1} & & x_1 & \\
x_2 & & \frac{x_1x_3+1}{x_2} & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & x_2 & & \\
& x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & \frac{x_2+1}{x_3} & & x_3 & \\
1 & & 1 & & 1 & & 1 & &
\end{array}$$

Figure 1. The identity frieze $Id : \mathcal{A}(Q) \to \mathcal{A}(Q)$ for the type $\mathbb{A}_3$ quiver $Q = 1 \to 2 \leftarrow 3$.

Proposition 1

Let $\mathcal{F}$ be a positive unitary integral frieze. Then the cluster $\mathbf{x}$ such that $\mathcal{F}(\mathbf{x}) = (1, \dots, 1)$ is unique. If such $\mathbf{x}$ exists, then it is unique.

Proof: If u is a cluster variable not in a cluster $\mathbf{x}$, then the Laurent expansion of u in $\mathbf{x}$ has two or more terms.

Examples

- The identity frieze $Id : \mathcal{A}(Q) \to \mathcal{A}(Q)$.
- A frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Z}$ defined by fixing a cluster $\mathbf{x}$ and sending each cluster variable in $\mathbf{x}$ to 1.

$$\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
& 1 & & 3 & & 2 & & 1 \\
1 & & 2 & & 5 & & 1 & \\
& 1 & & 3 & & 2 & & 1 \\
1 & & 1 & & 1 & & 1 &
\end{array}$$

Figure 2. Setting $x_1{=}x_2{=}x_3{=}1$ produces a Conway – Coxeter frieze pattern.

II. Frieze Vectors

Fix a cluster $\mathbf{x} = (x_1, \dots, x_n)$.

- A vector $(a_1, \dots, a_n) \in \mathbb{Z}^n_{>0}$ can be used to define a frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Q}$ by defining $\mathcal{F}(x_i) = a_i$ for all $i = 1, \dots, n$.
- We say that $(a_1, \dots, a_n)$ is a ***positive frieze vector relative to*** $\mathbf{x}$ if $\mathcal{F}$ maps every cluster variable to a positive integer (as opposed to $\mathbb{Q}$).
- If $(a_1, \dots, a_n)$ determines a unitary frieze, we say that $(a_1, \dots, a_n)$ is a ***unitary*** frieze vector.

Theorem 2

Fix $\mathcal{A}(Q)$ and fix $\mathbf{x} = (x_1, \dots, x_n)$ an arbitrary cluster. Define

$$\phi : \{ \text{ unordered clusters } \} \to \{ \text{ positive unitary frieze vectors } \}$$
$$\mathbf{x}' = \{x'_1, \dots, x'_n\} \mapsto \phi(\mathbf{x}') = \mathcal{F}(\mathbf{x}) = (a_1, \dots, a_n)$$

where and $\mathcal{F}$ is the frieze defined by specializing the cluster variables in $\mathbf{x}'$ to 1. Then ϕ is a bijection.

Proof: Injectivity follows from Proposition 1. Surjectivity follows from the construction of ϕ.

Proposition 3

Let $(\mathbf{x}, Q)$ be an acyclic seed. Then a vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$ is a frieze vector relative to $\mathbf{x}$ iff a_k divides

$$\prod_{k \to j} x_j + \prod_{k \leftarrow j} x_j$$

for all $k = 1, \dots, n$.

- A vector $(a_1, a_2, a_3) \in \mathbb{Z}^3_{>0}$ is a positive frieze vector relative to the cluster

$$(x_1, x_2, x_3), Q = 1 \to 2 \leftarrow 3$$

iff

$$\frac{a_2 + 1}{a_1}, \frac{a_1 a_3 + 1}{a_2}, \frac{a_2 + 1}{a_3}$$

are integers.
- Given any type $\mathbb{A}_3$ quiver, there are 14 positive frieze vectors. The values of these vectors depend on the quiver.

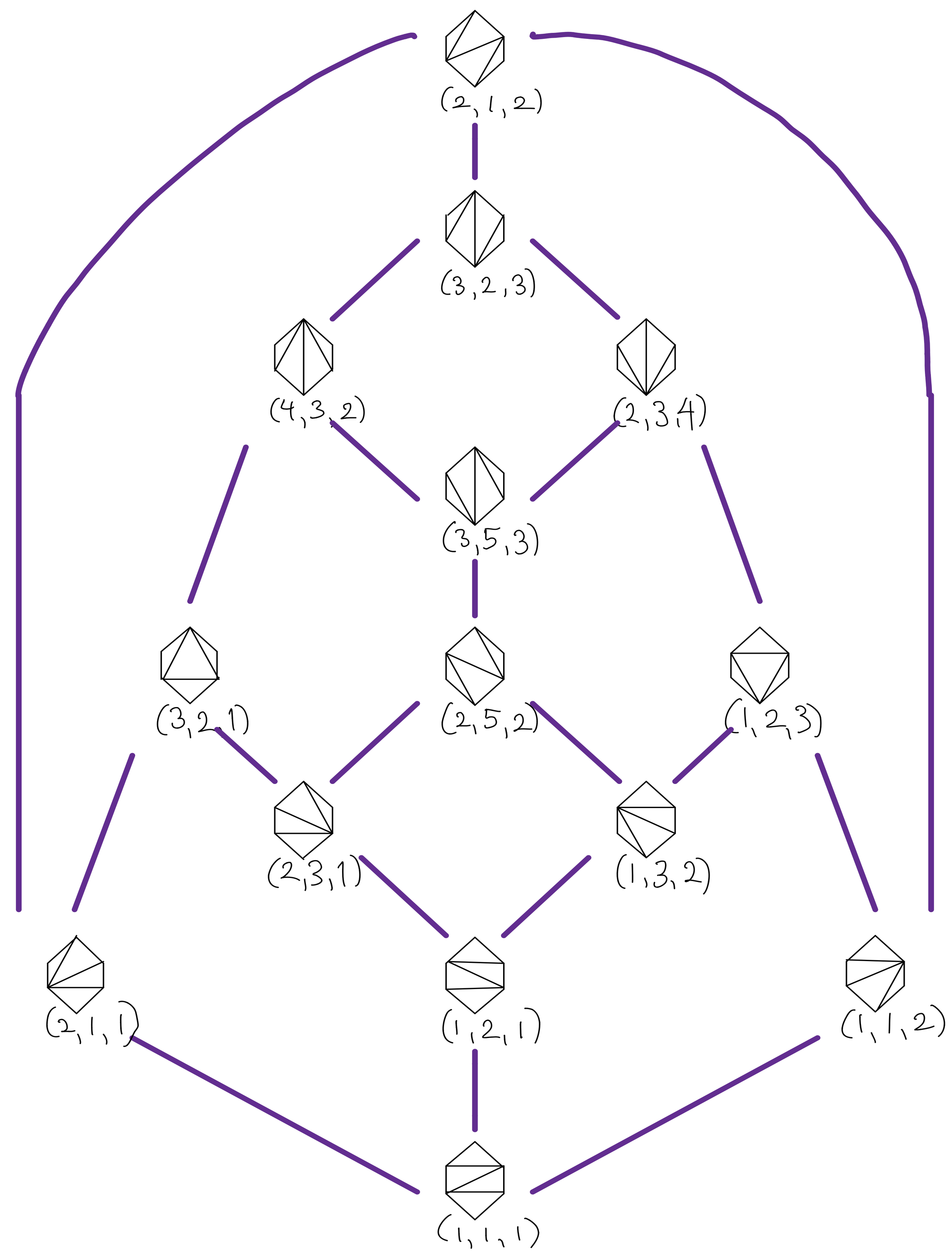

Figure 3. Positive Frieze vectors relative to $Q = 1 \to 2 \leftarrow 3$.

III. Type $\widetilde{\mathbb{A}}$ friezes

Lemma 4

Let $\mathcal{F}$ be a positive frieze of type $\widetilde{\mathbb{A}}_{p,q}$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a cluster such that $\mathcal{F}(x) = 1$ for each regular (i.e. peripheral) cluster variable. Let k be such that $\mathcal{F}(x_k) \geq \mathcal{F}(x_j)$ for all j, and suppose that $\mathcal{F}(x_k) > 1$. Then $\mathcal{F}(\mu_k(x_k)) < \mathcal{F}(x_k)$ and if $\mu_k(x_k)$ a regular cluster variable then $\mathcal{F}(\mu_k(x_k)) = 1$.

Theorem 5

All type $\widetilde{\mathbb{A}}_{p,q}$ friezes are unitary.

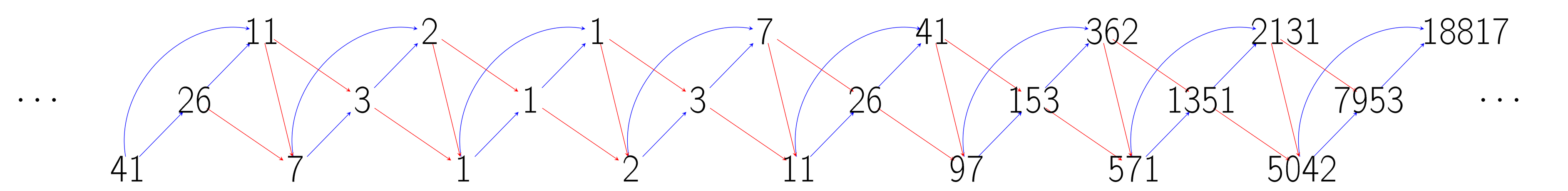

Figure 4. An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

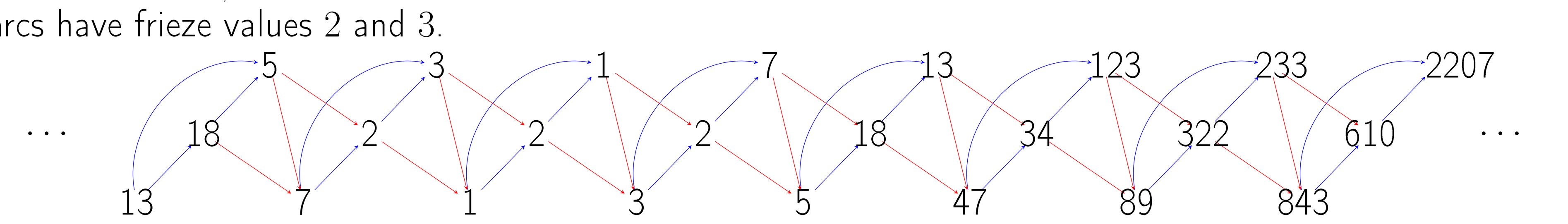

Figure 5. An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Remark: All positive integral friezes of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ are unitary, but there are non-unitary positive integral friezes of type $\mathbb{D}$, $\widetilde{\mathbb{D}}$, $\mathbb{E}$, and $\widetilde{\mathbb{E}}$.