Box-Ball Systems and Robinson–Schensted–Knuth Tableaux

University of Oklahoma ARTS Seminar

Friday, April 23, 2021

UConn Math REU 2020
Ben Drucker, Eli Garcia, Aubrey Rumbolt, and Rose Silver

Motivation: Soliton waves

- At time $-\infty$, soliton waves are traveling through space at different speeds, not minding each other.
- At some time, they begin to collide with one another, causing interference, and for a while you have a mess.
- But eventually by time $+\infty$ the interference sorts itself out, and the solitons continue on their way as if it hadn't happened.

Box-Ball System: Example—$\pi = 452361$

Start with an initial configuration $\pi = \pi_1\pi_2\pi_3...\pi_k$, where π is a permutation.

Step 1: Write the permutation on a strip of infinite boxes:

$t = 0$

4	5	2	3	6	1							...

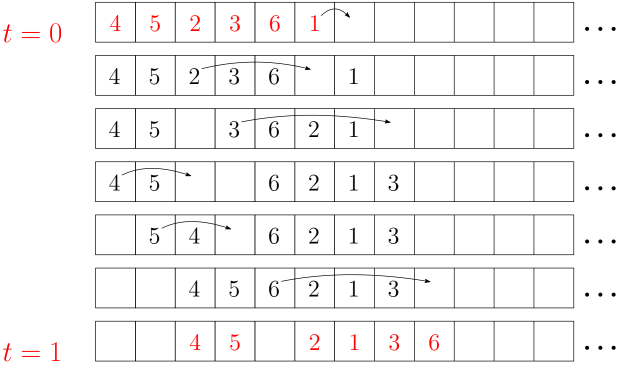
Step 2: To complete a box-ball move, let each number (or "ball") jump to the next available spot (or "box") to the right. First move 1, then move 2, and so on.

$t = 0$

4	5	2	3	6	1 ↷							...
4	5	2	3	6		1						...

Box-Ball System: Example—$\pi = 452361$

Step 3: Continue moving numbers from smallest to largest to their nearest available spots until every number in the permutation has been moved.

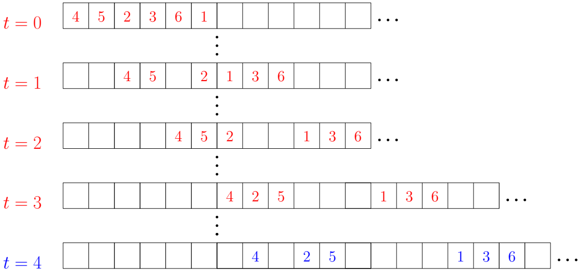


We are now at the $t = 1$ state and we have completed one BBS move.

Box-Ball System: Example—$\pi = 452361$

Step 4: Continue making BBS moves.

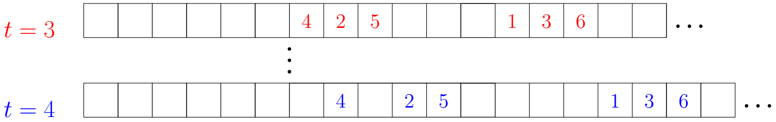
(Here, 4 moves are shown).



Box-Ball System: Soliton Decomposition

After a finite number of moves, the system reaches a *steady state* where:

- blocks of increasing sequences (or *solitons*) move together at a speed equal to their length.
- the sizes of the solitons are weakly increasing from left to right
- order of the solitons remain unchanged



Box-Ball System: Example—$\pi = 452361$

Step 5: After reaching steady state, create a *soliton decomposition* diagram $\mathrm{SD}(\pi)$ by stacking solitons from right to left.

$t = 4$

							4		2	5					1	3	6		...

The shape of the diagram always forms a partition (weakly decreasing sequence of positive integers):

Soliton decomposition $\mathrm{SD}(\pi) =$

1	3	6
2	5	
4		

with shape $(3, 2, 1)$.

REU Questions.

When does a permutation reach its steady state?

How many permutations in S_n first reach its steady state at a given time t?

Tableaux

Definition. (Young Tableaux)

- A *tableau* is an arrangement of numbers $\{1, 2, \ldots, n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if the rows and columns are increasing sequences.
- The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

Example. (Standard Young Tableau)

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$ is a standard tableau. Its reading word is 53412.
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ is a nonstandard tableau.

REU Question.

When is a soliton decomposition standard?

Robinson-Schensted insertion algorithm

The *Robinson-Schensted (RS) insertion algorithm* is a bijection from permutations π to pairs of standard tableaux $(P(\pi), Q(\pi))$ called the insertion tableau and the recording tableau of π.
Example: $\pi = 452361$

$$P(\pi) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, \quad Q(\pi) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

Fact: Let r be the reading word of a standard tableau T. Then $P(\pi) = T$.

P-tableau vs soliton decomposition

REU Question.

For what permutations π do we have $P(\pi) = SD(\pi)$?

Lemma

- A permutation r is the reading word of a standard tableau T if and only if it reaches its soliton decomposition at $t = 0$.
- In particular, if r is the reading word of T, then $P(r) = T = SD(\pi)$.

Theorem

The following are equivalent:

1. $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$.
2. $\mathrm{SD}(\pi)$ is a standard tableau.
3. $\operatorname{sh} \mathrm{SD}(\pi) = \operatorname{sh} \mathrm{P}(\pi)$.

Knuth Relations

Definition

Suppose π, $w \in S_n$ and $x < y < z$.

- π and w differ by a Knuth relation of the **first kind** (K_1) if

$$\pi = x_1 ... yxz ... x_n \text{ and } w = x_1 ... yzx ... x_n$$

- π and w differ by a Knuth relation of the **second kind** (K_2) if

$$\pi = x_1 ... xzy ... x_n \text{ and } w = x_1 ... zxy ... x_n$$

- π and w differ by Knuth relations of **both kinds** (K_B) if

$$\pi = x_1 ... y_1 xzy_2 ... x_n \text{ and } w = x_1 ... y_1 zxy_2 ... x_n$$

for $x < y_1, y_2 < z$

Example

$\mathbf{326}154 \sim^{K_1} \mathbf{362}154$
$362\mathbf{154} \sim^{K_B} 362\mathbf{514}$

Facts (Knuth)

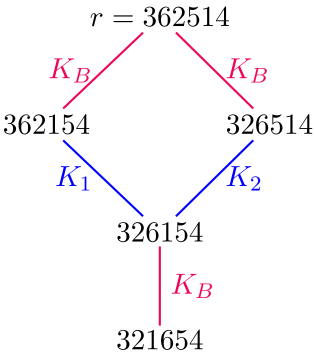
- There is a path of Knuth moves from π to the reading word of $P(\pi)$.
- Two permutations have the same RS insertion tableau if and only if they are related by a sequence of Knuth moves.

Example

The Knuth equivalence class of $r = 362514$, the reading word of the tableau

1	4
2	5
3	6

.



Soliton decompositions and Knuth moves

The soliton decomposition is preserved by non-K_B Knuth moves, but one K_B move changes the soliton decomposition.

Theorems

Let r denote the reading word of $P(\pi)$.

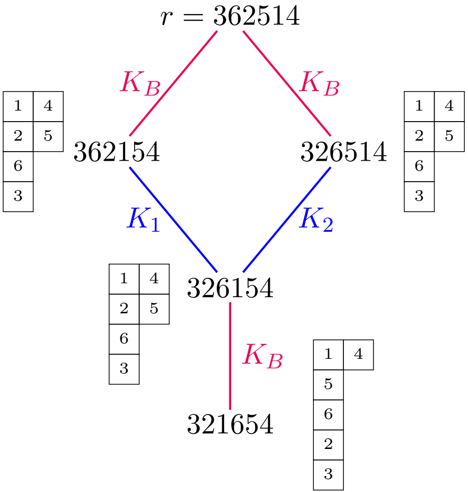
- If there exists a path of *non-*K_B Knuth moves from π to r, then $\mathrm{SD}(\pi) = P(\pi)$.
- If there exists a path from π to r containing an *odd* number of K_B moves, then $\mathrm{SD}(\pi) \neq P(\pi)$.

Soliton decompositions in a Knuth equivalence class

The permutation $r = 362514$ is the reading word of the tableau

1	4
2	5
3	6

.



Q-tableau and steady-state value

Theorem

If $n \geq 5$ and a permutation w in S_5 has recording tableau

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1}\end{array},$$

then w first reaches a steady-state configuration at time $n-3$.

The time when w first reaches steady state is called the *steady-state value* of w.

Conjecture

1. All other permutations in S_n have steady-state value less than $n-3$; i.e., the set of permutations with steady-state value $n-3$ are counted by standard Young tableaux of shape $(n-3, 2, 1)$.
2. If two permutations have the same recording tableau, they have the same steady-state value.

Thank You!