INFINITE FRIEZES & BRACELETS

1. Conway-Coxeter (finite) friezes

Row of 0's
Row of 1's
Positive integers
SL_2-rule

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc = 1$$

$$\begin{array}{ccccccccccccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2\ \cdots \\
\cdots & & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 \\
 & & & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 \\
 & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

bounded below by a row of 1s

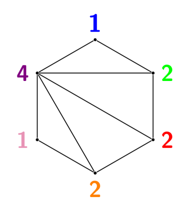
Thm (Conway & Coxeter 1970s)

$\left\{ \begin{array}{l} \text{Conway-Coxeter friezes} \\ \text{with } n \text{ nontrivial rows} \end{array} \right\} \longleftrightarrow \left\{ \text{triangulations of } (n+3)\text{-gon} \right\}$

Ex. $n=3$

1 4 1 2 2 2

quiddity sequence



6-gon

Thm (Propp + REACH (REU) 2001, Caldero & Chapoton 2004)

Finite friezes $\Rightarrow$ cluster algebras type A_n

Positive integer entries $\longleftrightarrow$ # terms in Laurent polynomial expansions of cluster variables

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & \\
 & x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & \frac{x_2+1}{x_1} & & x_1 & \\
x_2 & & \frac{x_1x_3+1}{x_2} & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & x_2 & & \\
 & x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & \frac{x_2+1}{x_3} & & x_3 & \\
1 & & 1 & & 1 & & 1 & &
\end{array}$$

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 3 & & 2 & & 1 & \\
1 & & 2 & & 5 & & 1 & & \\
 & 1 & & 3 & & 2 & & 1 & \\
1 & & 1 & & 1 & & 1 & &
\end{array}$$

2. Infinite friezes

Same rule, but remove the bottom boundary condition (Tschabold 2015)

Ex: periodic with $n=5$

Row of 0's
Row of 1's
Positive integers

SL_2-rule

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

$ad - bc = 1$

0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 1 ...
4 1 2 3 2 4 1 2 3 2 4 1
3 1 5 5 7 3 1 5 5 7 3 1
2 2 8 17 5 2 2 8 17 5 2 2
3 3 27 12 3 3 3 27 12 3 3
... 4 10 19 7 4 4 10 19 7 4 4
13 7 11 9 5 13 7 11 9 5
9 4 14 11 16 9 4 14 11 16
5 5 17 35 11 5 5 17 35
6 6 54 24 6 6 54 24 6
⋮

Classification of Infinite Friezes (Baur, Parsons, Tschabold 2015)

Every infinite periodic frieze corresponds to a triangulation of an annulus or once-punctured disk

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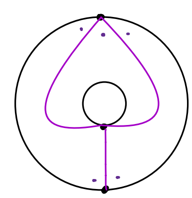
1 1 1 1 1 1
2 3 2 3 2 3
5 5 5 5 5 5
8 12 8 12 8 12
19 19 19 19 19 19
30 45 30 45 30 45
71 71 71 71 71 71
112 168 112 168 112 168
265 265 265 265 265 265

Ex 1: annulus



quiddity sequence: 2 3

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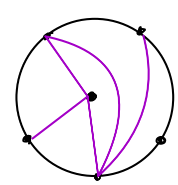
1 1 1 1 1 1 1 1 1 1 1 1
4 1 2 3 2 4 1 2 3 2 4 1
3 1 5 5 7 3 1 5 5 7 3 1
2 2 8 17 5 2 2 8 17 5 2 2
3 3 27 12 3 3 3 27 12 3 3
4 10 19 7 4 4 10 19 7 4 4
13 7 11 9 5 13 7 11 9 5
9 4 14 11 16 9 4 14 11 16
5 5 17 35 11 5 5 17 35
6 6 54 24 6 6 54 24 6

Ex 2: punctured disk



quiddity sequence: 412324

Later: infinite frieze with cluster algebra elements

3. Cluster algebras (Fomin – Zelevinsky, 2001)

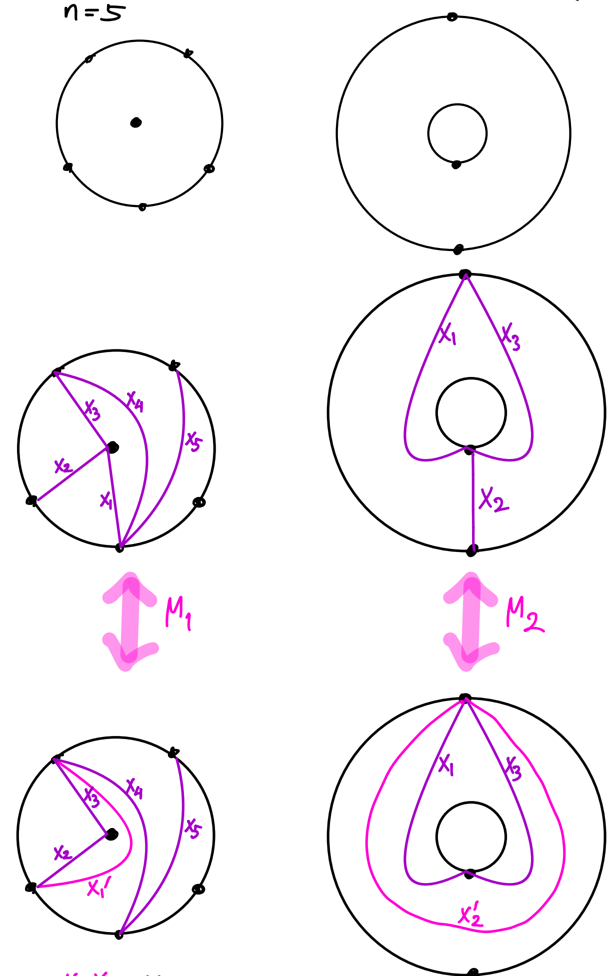
Idea: A cluster algebra is a subring $\mathcal{A}$ of $\mathbb{Q}(x_1, \dots, x_n)$

- generated by cluster variables.
- Start with n initial cluster variables + some data, then compute all cluster variables iteratively

Cluster algebras from surfaces (Fomin-Shapiro-Thurston, 2006)

D_n — once-punctured disk, $n=5$

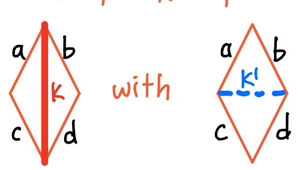
$\tilde{A}_{p,q}$ — annulus w/ $p+q$ marked points on the boundary, $p,q = 1,2$



An arc is an internal curve between marked points

A triangulation is a maximal collection of non-crossing arcs

A flip M_k replaces



Ptolemy rule

$$k' = \frac{ad+bc}{k}$$

$$x_1' = \frac{x_2x_4 + x_3}{x_1}$$

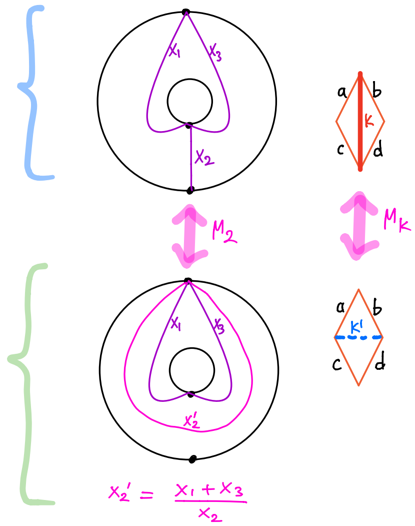
$$x_2' = \frac{x_1 + x_3}{x_2}$$

How to construct cluster variables

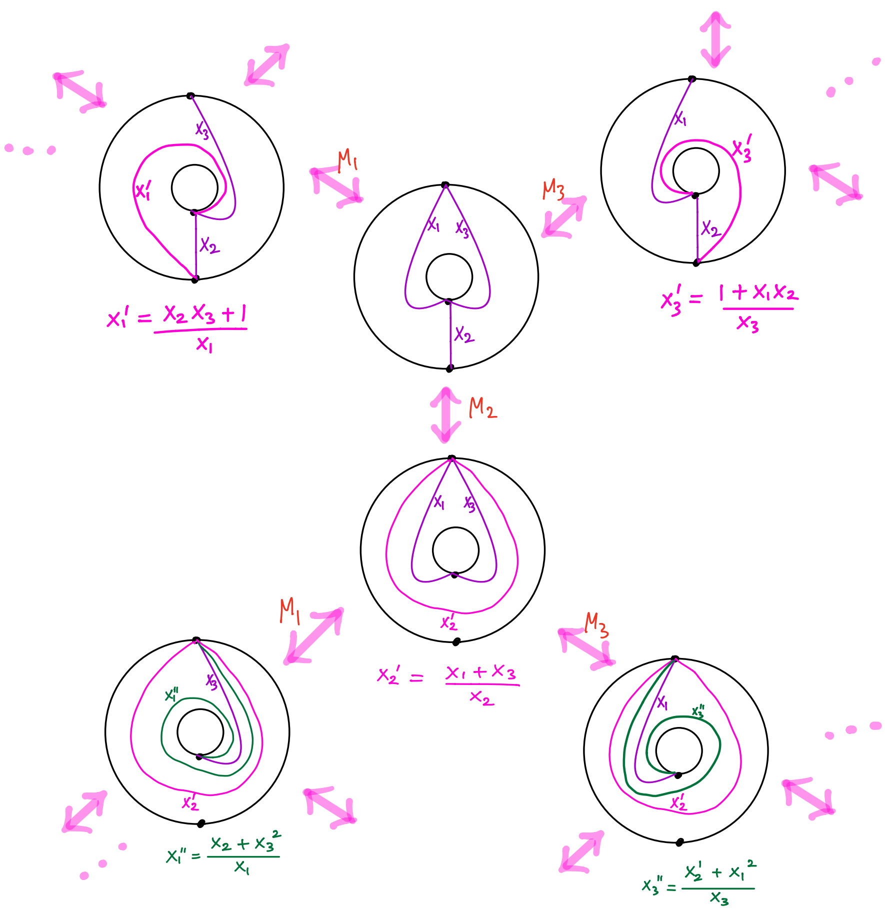
initial cluster $\{x_1, x_2, x_3\}$

Mutation at x_2

a new cluster $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$



Repeat this mutation process to produce all clusters



$$\{\text{cluster variables}\} = \bigcup_{\text{all clusters } \mathcal{X}} \{\text{elements of } \mathcal{X}\}$$

Laurent phenomenon & positivity: Surprisingly, every cluster variable is a Laurent polynomial w/ positive coefficients in the initial cluster $\qquad \dfrac{\text{polynomial}}{\text{monomial}}$

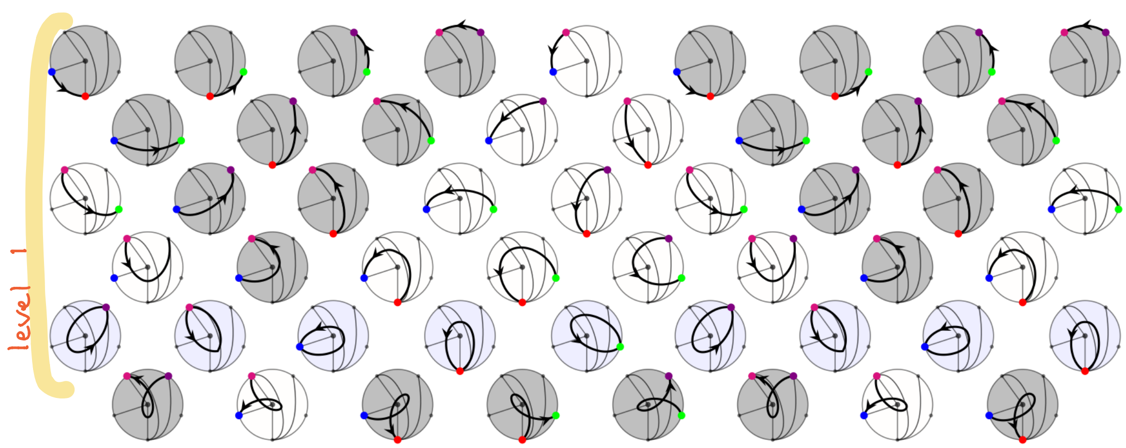
For cluster algebras from surfaces:

arcs $\longleftrightarrow$ cluster variables

generalized arcs (self-crossing is allowed) $\longrightarrow$ cluster algebra elements which are Laurent polynomials with positive coefficients

4. Infinite frieze of positive Laurent polynomials

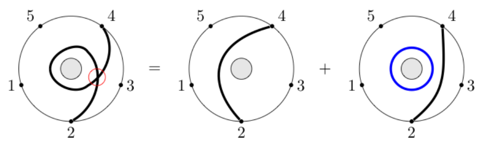
The cluster algebra elements corresponding to generalized arcs between the same boundary of an annulus or a punctured disk form an infinite frieze.



An infinite frieze of elements of the cluster algebra corresponding to peripheral curves in a punctured disk.

<u>Why?</u> The self-intersecting arcs correspond to elements of $\mathcal{A}$ via skein relation (Musiker & Williams 2011)

Example:



Resolving a self-crossing.

- When the variables are specialized to 1, we recover the integer frieze pattern. When specialized to nonzero numbers, we get an infinite frieze pattern with nonzero entries.

1 1 1 1 1 1 1 1 1 1 1
4 **1** 2 3 2 4 **1** 2 3 2 4
3 **1** 5 5 7 3 **1** 5 5 7
2 2 8 17 5 2 2 8 17 5
3 3 27 12 3 3 3 27 12

---

4 10 19 7 4 4 10 19 7
13 7 11 9 5 13 7 11
9 4 14 11 16 9 4 14
5 5 17 35 11 5 5
6 6 54 24 6 6 6

Divide rows into entries

Level 1 consists of curves with 0 self-crossings

Level 2 consists of curves with 1 self-crossing

⋮

Level k consists of curves with $k-1$ self-crossings

An infinite frieze pattern

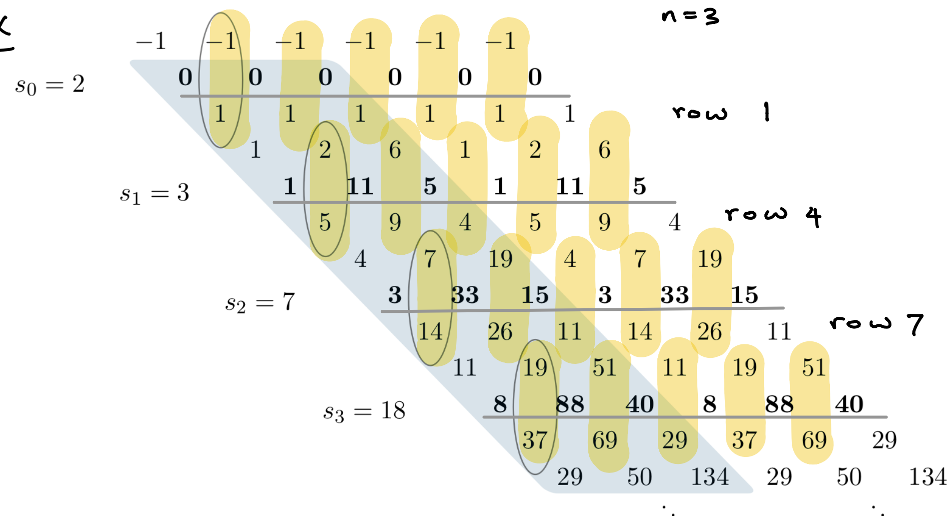
periodic with $n=5$

```
       1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
         4   1   2   3   2   4   1   2   3   2   4   1   2   3   2
           3   1   5   5   7   3   1   5   5   7   3   1   5   5   7
             2   2   8  17   5   2   2   8  17   5   2   2   8  17
Level 1        3   3  27  12   3   3   3  27  12   3   3   3  27  12
               -----------------------------------------------------
                 4  10  19   7   4   4  10  19   7   4   4  10  19
                  13   7  11   9   5  13   7  11   9   5  13   7  11
                     9   4  14  11  16   9   4  14  11  16   9   4
Level 2                5   5  17  35  11   5   5  17  35  11   5   5
                         6   6  54  24   6   6   6  54  24   6   6
                         -------------------------------------------
                           7  19  37  13   7   7  19  37  13   7   7
Level 3                     22  13  20  15   8  22  13  20  15   8
                              15   7  23  17  25  15   7  23  17  25
                                 8   8  26  53  17   8   8  26  53
                                   9   9  81  36   9   9   9  81  36
                                   ---------------------------------
                                    10  28  55  19  10  10  28  55
Level 4                               31  19  29  21  11  31  19  29
                                        21  10  32  23  34  21
```

each level has $n=5$ rows

5. Growth coefficients

Ex



[Baur, Fellner, Parsons, Tschabold 2016]

- In an n-periodic infinite frieze of positive integers, the difference between the entries in rows $(nk+1)$ & $(nk-1)$ (on the same column) is constant.

- The constants s_k's satisfy the normalized Chebyshev polynomials:

$$s_k = T_k(s_1) \quad \forall\ k$$

where the normalized Chebyshev polynomial is defined by

$$T_0(x) := 2$$

$$T_1(x) := x$$

$$T_k(x) := x\, T_{k-1}(x) - T_{k-2}(x)$$

6. Growth coefficients and bracelets

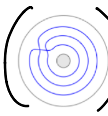
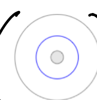
[Musiker - Schiffler - Williams 2011]

The element $x(Brac_k) \in \mathcal{A}$ associated to a bracelet which crosses itself $k-1$ times is an important element.

Bracelets $Brac_1$, $Brac_2$, and $Brac_3$.

- Certain products of the cluster variables and the bracelets form a nice basis of $\mathcal{A}$ called the bracelets basis.
- The elements $x(Brac_k)$ satisfy the normalized Chebyshev polynomials

$$x(Brac_k) = T_k(Brac_1)$$

Ex. $x\Big($  $\Big) = T_3\Big($  $\Big)$

[G., Musiker, Vogel 2016]

In the frieze of Laurent polynomials, the "jump" between level k & $k+1$ is the cluster algebra element which corresponds to the bracelet which crosses itself $k-1$ times.

The constant s_k = # terms in the Laurent expansion of $x(Brac_k)$

In punctured disk case, this growth factor is always 2.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

4 1 2 3 2 4 1 2 3 2 4 1 2 3 2 4 1 2 3

3 1 5 5 7 3 1 5 5 7 3 1 5 5 7 3 1 5

2 2 8 17 5 2 2 8 17 5 2 2 8 17 5 2 2 8

3 3 27 12 3 3 3 27 12 3 3 3 27 12 3 3 3 2

---

4 10 19 7 4 4 10 19 7 4 4 10 19 7 4 4 10

13 7 11 9 5 13 7 11 9 5 13 7 11 9 5 13

9 4 14 11 16 9 4 14 11 16 9 4 14 11 16 9

5 5 17 35 11 5 5 17 35 11 5 5 17 35 11

6 6 54 24 6 6 6 54 24 6 6 6 54 24 6

---

7 19 37 13 7 7 19 37 13 7 7 19 37 13

22 13 20 15 8 22 13 20 15 8 22 13 20 15

15 7 23 17 25 15 7 23 17 25 15 7 23 1

8 8 26 53 17 8 8 26 53 17 8 8 26

9 9 81 36 9 9 9 81 36 9 9 9 8

---

10 28 55 19 10 10 28 55 19 10 10 28

31 19 29 21 11 31 19 29 21 11 31 1

21 10 32 23 34 21 10 32 23 34 21

11 11 35 71 23 11 11 35 71 23 1

12 12 108 48 12 12 12 108 48 12

---

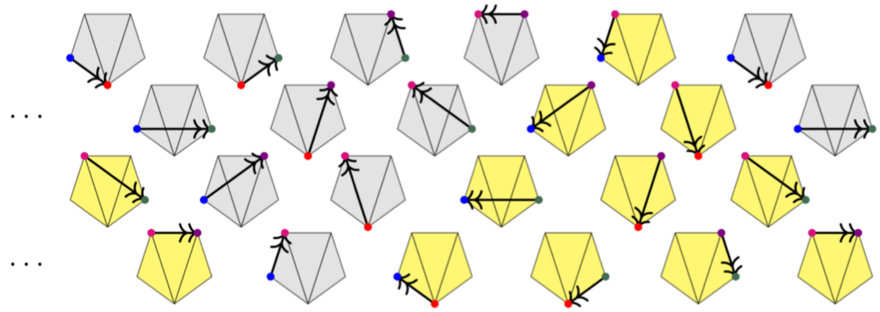
13 37 73 25 13 13 37 73 25 1

40 25 38 27 14 40 25 38

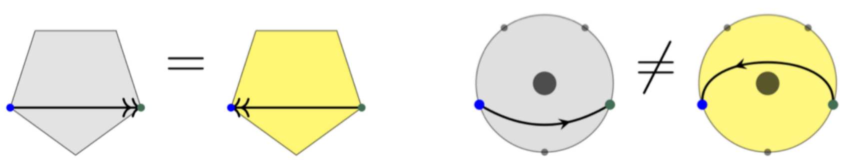
For the punctured disk, every bracelet corresponds to the integer 2

6. Complement symmetry

A Conway-Coxeter frieze is invariant under a glide reflection

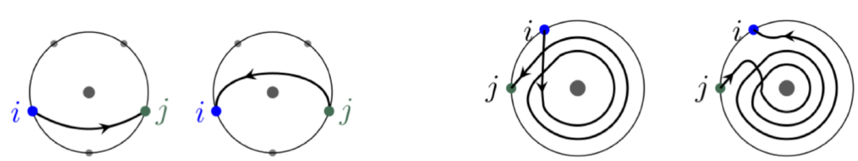


In a polygon vs a punctured disk/annulus



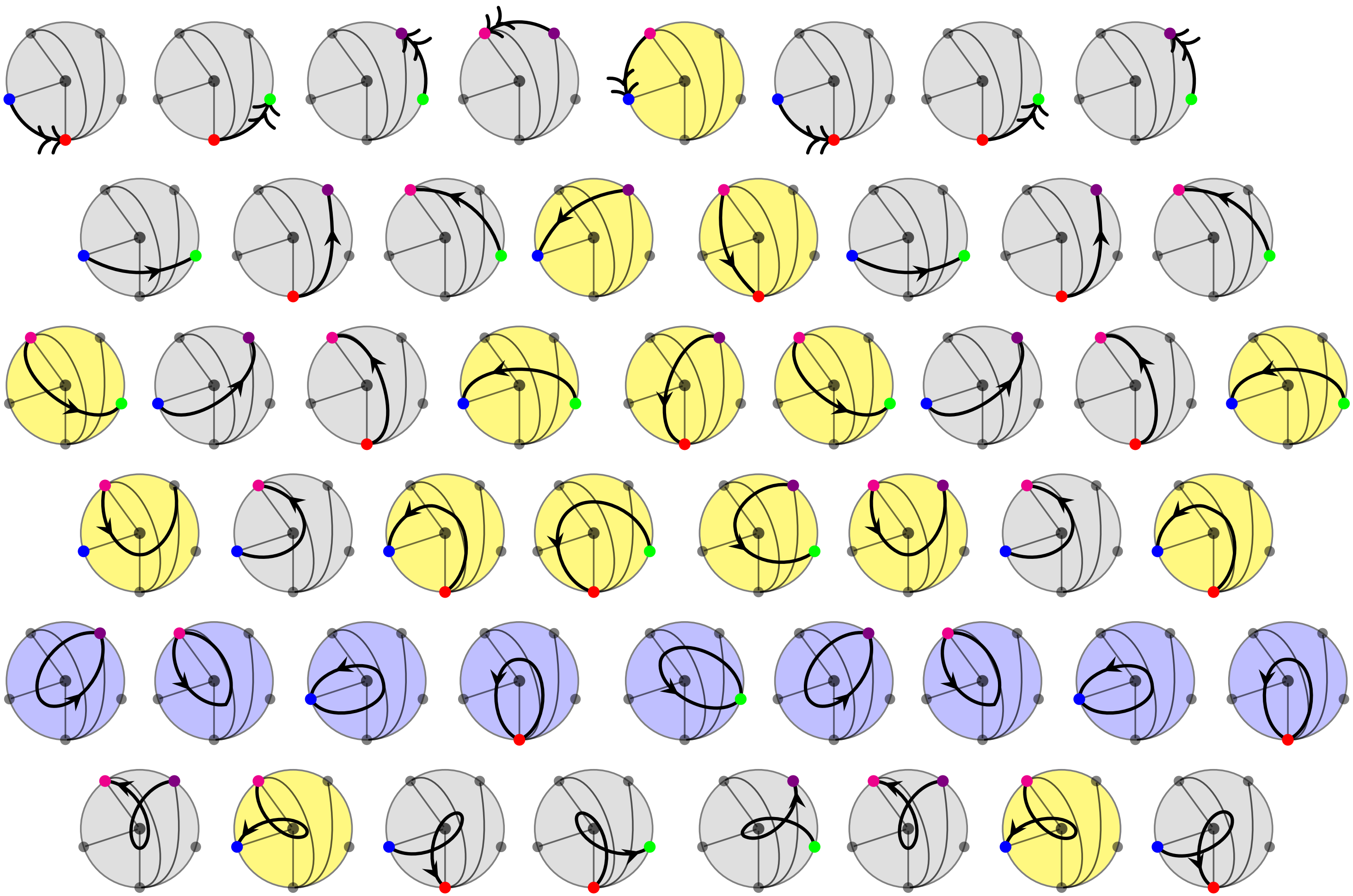
No glide reflection symmetry for infinite friezes, but there is a "complement symmetry"

Def Let $i < j$ and let γ_k be the arc from i to j with $k-1$ self-crossings. The complementary arc γ_k^C of γ_k is the arc from j to i with $k-1$ self-crossings



γ_1 $\quad$ γ_1^C $\quad$ γ_3 $\quad$ γ_3^C

Complementary arcs in infinite friezes



...

Arithmetic progressions in frieze patterns from punctured disks (Tschabold)

1 1 1 1 1 **1** 1 1 1 1 **1** 1 1 1 1 **1**

4 1 2 3 2 4 1 2 3 2 4 1 2 3 2

3 **1** 5 5 7 3 **1** 5 5 7 3 **1** 5 5 7

2 2 8 17 5 2 2 8 17 5 2 2 8 17

3 3 27 **12** 3 3 3 27 **12** 3 3 3 27 **12**

---

4 10 19 7 4 **4** 10 19 7 4 **4** 10 19

13 7 11 9 5 13 7 11 9 5 13 7 11

9 **4** 14 11 16 9 **4** 14 11 16 9 **4**

5 5 17 35 11 5 5 17 35 11 5 5

6 6 54 **24** 6 6 6 54 **24** 6 6

---

7 19 37 13 7 **7** 19 37 13 7 **7**

22 13 20 15 8 22 13 20 15 8

15 **7** 23 17 25 15 **7** 23 17 25

8 8 26 53 17 8 8 26 53

9 9 81 **36** 9 9 9 81 **36**

---

10 28 55 19 10 **10** 28 55

31 19 29 21 11 31 19

Geometric interpretation of the arithmetic progression

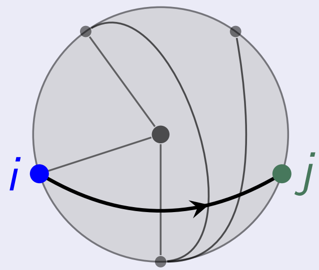
Proposition (G., Musiker, Vogel)

The arc from vertex blue to vertex green with k self-intersections

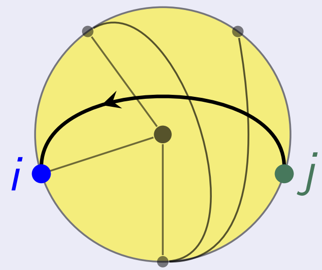
$$=$$

the arc from vertex blue to vertex green with $k-1$ self-intersections

$+$



$+$



Proof: Progression formulas and induction.

Progression formulas

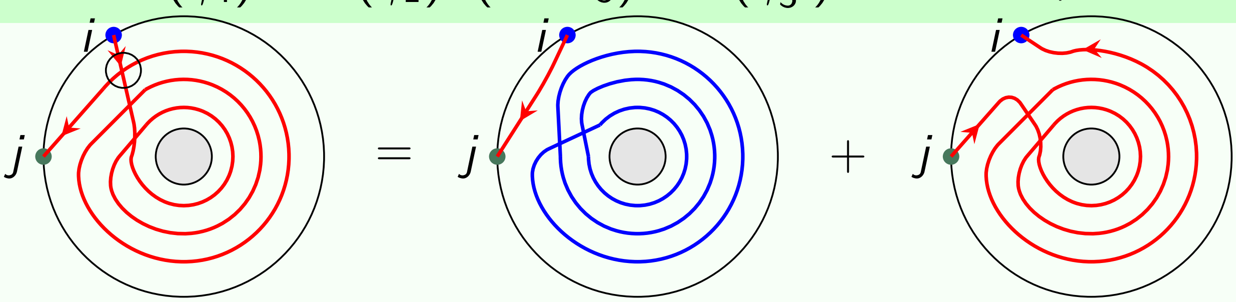
Theorem (G., Musiker, and Vogel)

Let γ_1 be an arc starting and finishing at vertices i and j. For $k = 1, 2, \ldots$ and $1 \leq m \leq k-1$, we have

$$x(\gamma_k) = x(\gamma_m)x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}), \text{ where:}$$

- *for $r \geq 0$, γ^C_{-r} is the curve γ_{r+1} with a kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$, and*
- *a* **bracelet** *$Brac_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, creating $(k-1)$ self-crossings.*

$$x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma^C_3) \text{ for } k = 4,\ m = 1$$



Thank
you!

Infinite friezes of cluster algebras from surfaces

Emily Gunawan, Gregg Musiker and Hannah Vogel

Cluster algebras from surfaces

A **cluster algebra** (Fomin – Zelevinsky 2000) is a subring of $\mathbb{Q}(x_1, \dots, x_n)$ with a distinguished set of generators, called **cluster variables**. Cluster variables are produced by an iterative process which can be read off from a directed graph or a skew-symmetric matrix.

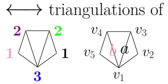
A **cluster algebra from a surface** (Fomin – Shapiro – Thurston 2006) is modeled by a Riemann surface S & marked points.

- arcs (with no self-crossing) $\longleftrightarrow$ cluster variables.
- Cluster algebras of type A, $\tilde{A}$, D, and $\tilde{D}$ arise from polygons, annuli, once-punctured polygons, and twice-punctured polygons, respectively.

Finite frieze patterns

A (Conway – Coxeter) **frieze pattern** is an array such that the top row is a row of 1s and every diamond $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ satisfies $ad - bc = 1$.

Theorem (Conway – Coxeter 1970s): Finite frieze patterns $\longleftrightarrow$ triangulations of polygons.



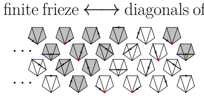
1 1 1 1 1 1 1 1 ⋯

Row 2 ⋯ 3 1 2 2 1 3 1

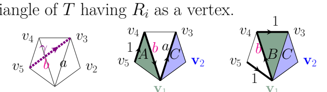
2 2 1 3 1 2 2 ⋯

⋯ 1 1 1 1 1 1 1 1

Theorem (Broline – Crowe – Isaacs 1970s): Entries of a finite frieze $\longleftrightarrow$ diagonals of a polygon.



Definition: Let T be a triangulation and γ an arc. Let $R_1, R_2, \dots, R_r$ be the boundary vertices to the right of γ. A **BCI tuple** for γ is a pairwise distinct r-tuple $(t_1, \dots, t_r)$ such that the i-th entry t_i is a triangle of T having R_i as a vertex.



T and γ — $(t_1, t_2) = (A, C)$, $1\, b^{-1}\, a$ + $(t_1, t_2) = (B, C)$, $1\, b^{-1}\, 1$

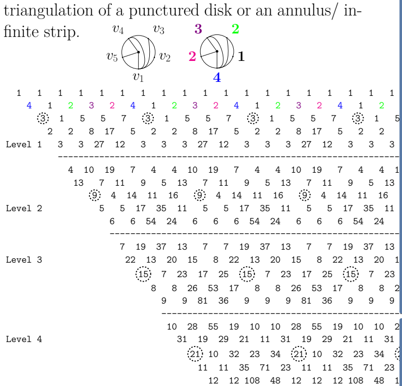
$x_\gamma = \frac{a}{b} + \frac{1}{b}$

Theorem (Caldero – Chapoton 2006): The cluster variables of a cluster algebra from a polygon (type A) form a finite frieze pattern. (Example: type A_2)

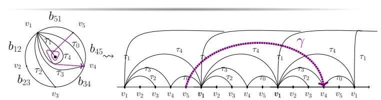
$$\begin{array}{ccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& \frac{1+a+b}{ab} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{ab} & & & \\
\cdots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{ab} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & \cdots \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Infinite frieze patterns

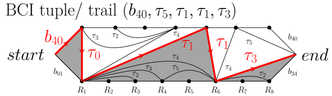
Theorem (Baur – Parsons – Tschabold 2015-2016): Every *infinite* frieze arises from a triangulation of a punctured disk or an annulus/ infinite strip.



Generalized arcs to cluster algebra elements



A BCI tuple/ trail $(b_{40}, \tau_5, \tau_1, \tau_1, \tau_3)$



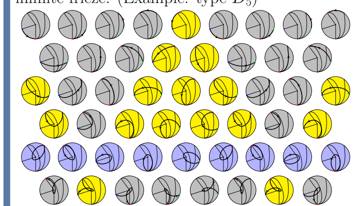
with weight $b_{40}\, x_0^{-1}\, x_1\, x_1^{-1}\, x_3 = \dfrac{1\, x_1\, x_3}{x_0\, x_1}$

The 11 BCI tuples correspond to the 11 terms of

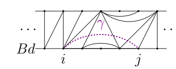
$$x_\gamma = \frac{\mathbf{x_0 x_1 x_4} + 2x_1x_3x_4 + 2x_0^2 + 4x_0x_3 + 2x_3^2}{x_0x_1x_4}$$

Infinite frieze

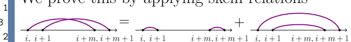
Theorem 1: The cluster algebra elements corresponding to generalized arcs between the same boundary of a punctured disk or annulus form an infinite frieze. (Example: type D_5)



Proof:
Given a triangulation T of the infinite strip, the arc $\gamma(i,j)$ from i to j on a boundary component Bd corresponds to the (i,j)-th entry in the infinite frieze arising from T.



We prove this by applying skein relations



Complementary arcs

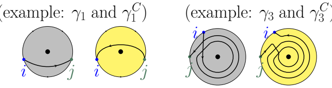
Definition: For $1 \le i, j \le n$ and $k \in \mathbb{N}$, let $\gamma_k(i,j)$ denote the arc that lifts to the infinite strip cover as follows:

$$\gamma_k(i,j) = \begin{cases} \gamma\,(i, j + (k-1)n) & \text{if } i < j \\ \gamma\,(i, j + kn) & \text{if } i \ge j \end{cases}.$$

That is, $\gamma_k(i,j)$ is the generalized arc that starts at i and finishes at j (possibly $i = j$) with $(k-1)$ self-intersections such that the boundary Bd is to the right of the curve as we trace it. Let the arc **complementary** to $\gamma_k = \gamma_k(i,j)$ be defined as

$$\gamma_k(i,j)^C = \begin{cases} \gamma\,(j, i + kn) & \text{if } i < j \\ \gamma\,(j, i + (k-1)n) & \text{if } i \ge j \end{cases}$$

(example: γ_1 and γ_1^C) (example: γ_3 and γ_3^C)



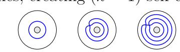
Progression formulas

Theorem 2: Let $\gamma_1 = \gamma$ be an arc from i to j (possibly $i = j$) or a boundary edge from i to $i+1$. For $k \in \mathbb{N}$ and $1 \le m \le k-1$, we have

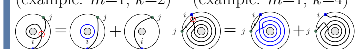
$$x(\gamma_k) = x(\gamma_m)x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}),$$

where:

- for $r \ge 0$, γ^C_{-r} is the curve γ_{r+1} with a contractible kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$, and
- a **bracelet** $Brac_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, creating $(k-1)$ self-crossings.



(example: m=1, k=2) (example: m=1, k=4)



$x(\gamma_2) = x(\gamma_1)x(Brac_1) + x(\gamma_1^C)$, $x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma_3^C)$

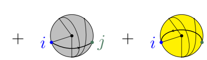
Arithmetic progression

Proposition 1: Let $\mathcal{F}$ be an infinite frieze from a punctured disk. Then

(the arc from i to j with k self-intersections)

=

(the arc from i to j with $k-1$ self-intersections)

+ 

Proof: Progression formulas and induction.

Complementary arc differences

Proposition 2: Let $\gamma_1 = \gamma$ be an arc from i to j (possibly $i = j$) or a boundary edge from i to $i+1$. Define $c_k := x(\gamma_k) - x(\gamma_k^C)$. Let $c_0 := c_1$. Then, for $k \ge 2$, we have

1. $c_k = (Brac_{k-1} - Brac_{k-2})c_1 + c_{k-2}$,
2. $c_k = c_1(1 + \Sigma_{i=0}^{k-1}(-1)^{i+\alpha}s_i)$, where $\alpha = 1$ if k is even and $\alpha = 0$ otherwise.

Note that, if $i = j$, then $c_1 = x(\gamma_1)$.

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