Cluster algebras and binary words

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Outline

- Bijection between binary subwords and order filters of a poset
- Bijection between binary subwords and perfect matchings of a snake graph (terms of a cluster variable)

Binary word

Definition

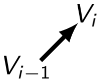
- A *binary word* is a finite sequence of letters belonging to $\{0, 1\}$. **In this talk, consider only words that start with 1.**
  - Example: 10100.
- A *subword* is a subsequence of a word.
  - Example: Some subwords of 10100 are the empty word, 11, 100, and itself.
  - Non-examples: 10001, 11000 are *not* subwords of 10100.
  - Note: Even though 1010 appears twice as a subsequence of 10100, we treat it as one subword.

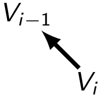
Subwords of 1011101100

1.	empty	22.	10110	43.	111010	64.	1111011
2.	1	23.	10111	44.	111011	65.	1111100
3.	10	24.	11000	45.	111100	66.	1111110
4.	11	25.	11010	46.	111101	67.	10101100
5.	100	26.	11011	47.	111110	68.	10110100
6.	101	27.	11100	48.	111111	69.	10110110
7.	110	28.	11101	49.	1001100	70.	10111000
8.	111	29.	11110	50.	1010100	71.	10111010
9.	1000	30.	11111	51.	1010110	72.	10111011
10.	1001	31.	100100	52.	1011000	73.	10111100
11.	1010	32.	100110	53.	1011010	74.	10111110
12.	1011	33.	101000	54.	1011011	75.	11101100
13.	1100	34.	101010	55.	1011100	76.	11110100
14.	1101	35.	101011	56.	1011101	77.	11110110
15.	1110	36.	101100	57.	1011110	78.	11111100
16.	1111	37.	101101	58.	1011111	79.	101101100
17.	10000	38.	101110	59.	1101100	80.	101110100
18.	10010	39.	101111	60.	1110100	81.	101110110
19.	10011	40.	110100	61.	1110110	82.	101111100
20.	10100	41.	110110	62.	1111000	83.	111101100
21.	10101	42.	111000	63.	1111010	84.	1011101100

Binary word to poset

Associate a subword $w = w_1 w_2 \ldots w_n$ to the Hasse diagram of a "line" poset with n elements $V_1, V_2, \ldots, V_n$ by assigning each w_i ($i \geq 2$) so that

1 corresponds to 

, and 0 corresponds to 

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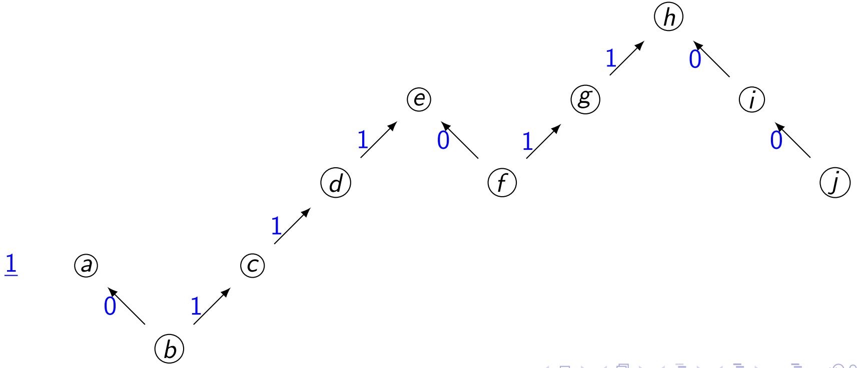
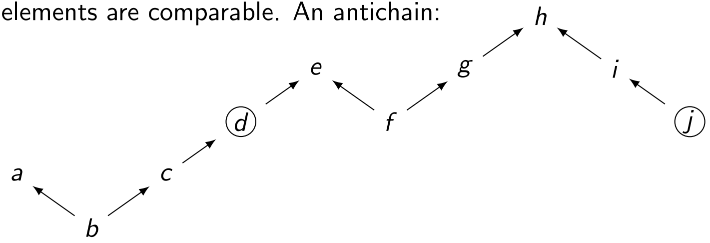
Example: $w = 1011101100$

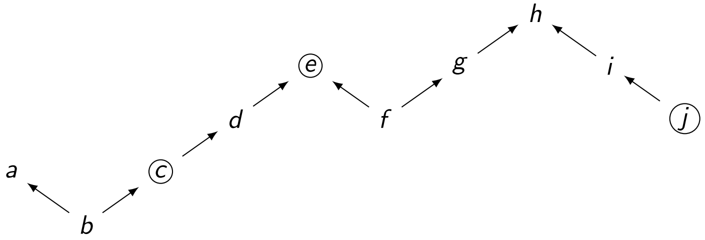


Antichain

An *antichain* of a poset P is a subset of P such that no 2 distinct elements are comparable. An antichain:



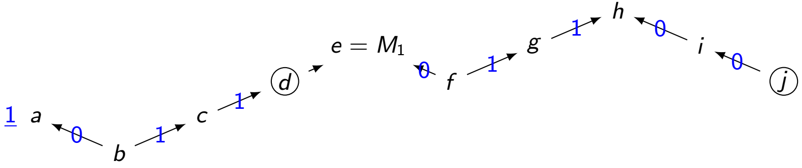
Not an antichain:



Bijection from antichains A to subwords s (G.)

The empty antichain is mapped to the empty word. Otherwise, map the antichain $A = \{A_1, A_2, \ldots, A_r\}$ to the following subword of w:

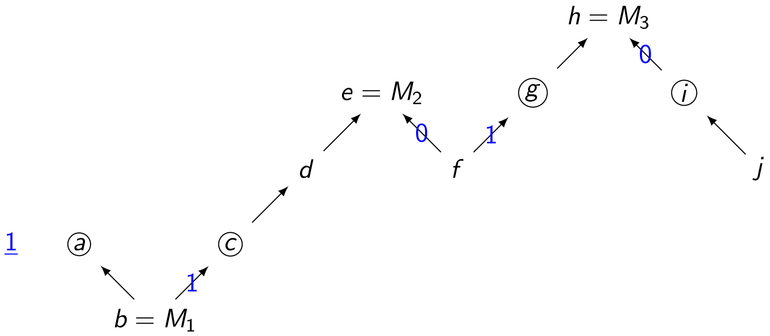
- 1 is the first letter. $s = 1$________
- The next letters are the (possibly empty) sequence of edge labels between the first element of P and A_1. $s = 1\boxed{011}$______



- If A contains only one element, we are done. Otherwise, jump to the first element M_1 appearing after A_1 which is either minimal or maximal. The elements of P between A_1 and M_1 (inclusive) are all comparable to A_1. Since A is an antichain, none of these are in A.
- Record the labels of the edges between M_1 and A_2. $s = 1011\boxed{01100}$
- Jump to the first element M_2 appearing after A_2 which is either minimal or maximal. Record the labels of the edges between M_2 and A_3, and so on.

Bijection from antichains A to subwords s (G.)

Another example:
The antichain $A = \{A_1 = ⓐ, A_2 = ⓒ, A_3 = ⓖ, A_4 = ⓘ\}$ of P
is mapped to the subword $s = \underline{1}\,\boxed{}\,\boxed{1}\,\boxed{01}\,\boxed{0}$ of w.

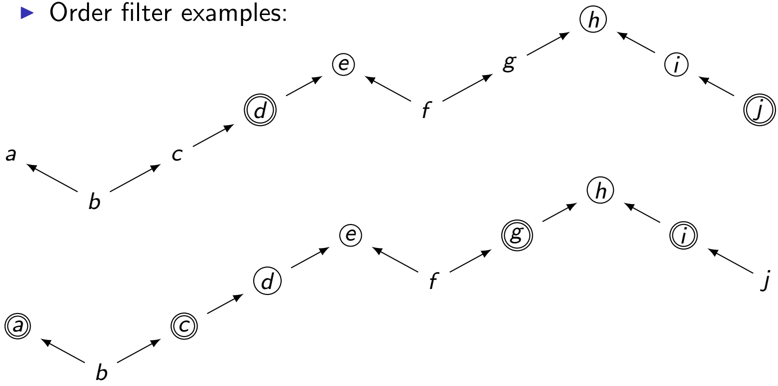


Order filter

Definition

An *order filter* (or dual order ideal) is a subset F of P such that if $t \in F$ and $s \geq t$, then $s \in F$.

- Fact: There is a one-to-one correspondence between antichains A and order filters F, where A is the set of minimal elements of F.
- Order filter examples:



Why order filter (as opposed to order ideal)?

Consider the quiver representation M over a field k

$$
\begin{tikzcd}[column sep=small]
 & & & & & & & k & & \\
 & & & & k & & k \arrow[ur, "id"] & & k \arrow[ul, "id"'] & \\
 & & & k \arrow[ur, "id"] & & k \arrow[ul, "id"'] \arrow[ur, "id"] & & & & k \arrow[ul, "id"'] \\
k & & k \arrow[ur, "id"] & & & & & & & \\
 & k \arrow[ul, "id"'] \arrow[ur, "id"] & & & & & & & &
\end{tikzcd}
$$

The subrepresentations of M correspond to the order filters of the poset.

Examples:

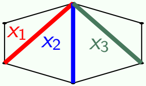
$$
\begin{tikzcd}[column sep=small]
 & & & & & & & k & & \\
 & & & & k & & 0 \arrow[ur, "0"] & & k \arrow[ul, "id"'] & \\
 & & & k \arrow[ur, "id"] & & 0 \arrow[ul, "0"'] \arrow[ur, "0"] & & & & k \arrow[ul, "id"'] \\
0 & & 0 \arrow[ur, "0"] & & & & & & & \\
 & 0 \arrow[ul, "0"'] \arrow[ur, "0"] & & & & & & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & & & & & & & k & & \\
 & & & & k & & k \arrow[ur, "id"] & & k \arrow[ul, "id"'] & \\
 & & & k \arrow[ur, "id"] & & 0 \arrow[ul, "0"'] \arrow[ur, "0"] & & & & 0 \arrow[ul, "0"'] \\
k & & k \arrow[ur, "id"] & & & & & & & \\
 & 0 \arrow[ul, "0"'] \arrow[ur, "0"] & & & & & & & &
\end{tikzcd}
$$

Cluster algebras (Fomin and Zelevinsky, 2000)

A **cluster algebra** is a subring of $\mathbb{Q}(x_1, \ldots, x_n)$ with a distinguished set of generators, called **cluster variables**.

Cluster algebras from surfaces (Fomin, Shapiro, and Thurston, 2006, etc.)

- A Riemann surface S + marked points gives rise to a cluster algebra.
- Starting from a triangulation and initial cluster variables $x_1, \ldots, x_n$, produce all the other cluster variables by an iterative process called mutation. 

- The cluster variables $\longleftrightarrow$ curves between marked points, called arcs.

Laurent Phenomenon (Fomin - Zelevinsky) and **positivity** (Lee - Schiffler, Gross - Hacking - Keel - Kontsevich, 2014, and special cases by others): Each cluster variable can be expressed as a Laurent polynomial in $\{x_1, \ldots, x_n\}$, that is, as

$$\frac{f(x_1, \ldots, x_n)}{x_1^{d_1} \ldots x_n^{d_n}},$$

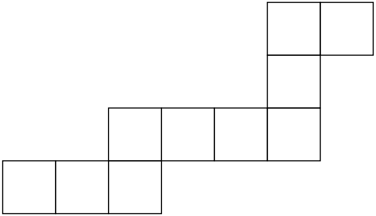
where f is a polynomial with positive coefficients.

Snake graphs

Definition

A *snake graph* is a connected sequence of square tiles . To build a snake graph, start with one tile, then glue a new tile to the north or the east of the previous tile.

Example of a snake graph with 10 tiles:



- History: Used by Musiker, Propp, Schiffler, and Williams to study positivity and bases of cluster algebras from surfaces (2005, 2009–10). The theory of abstract snake graph was developed further by Çanakçı, Lee, and Schiffler (2012–17).

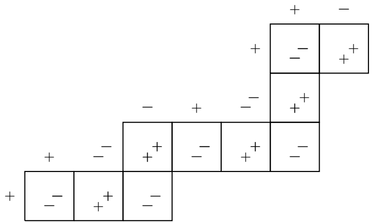
Sign function

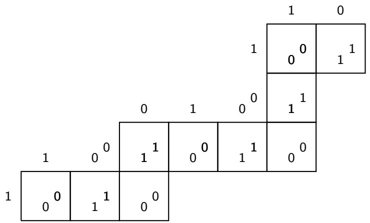
Definition (Çanakçı, Schiffler)

A *sign function* on a snake graph G is a map from the set of edges of G to $\{+, -\}$ such that, for every tile of G,

- the north and the west edges have the same sign,
- the south and the east edges have the same sign, and

- the sign on the north edge is opposite to the sign on the south edge.



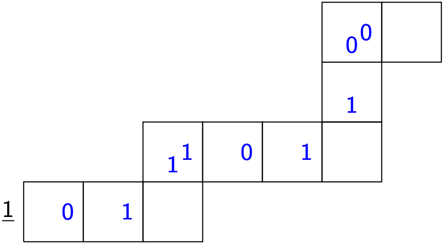


Note: There are two possible sign functions on G.

For convenience, we replace $+$ with 1 and $-$ with 0.

Sign Sequence

The sequence of signs of the interior edges corresponds to a binary word.

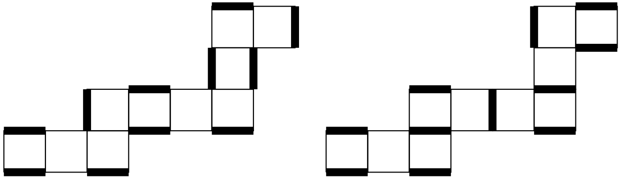


Example: the word $\underline{1}011101100$.

Perfect matchings

Definition

A *matching* of a graph G is a subset of non-adjacent edges of the graph. A *perfect matching* of G is a matching where every vertex of G is adjacent to exactly one edge of the matching.

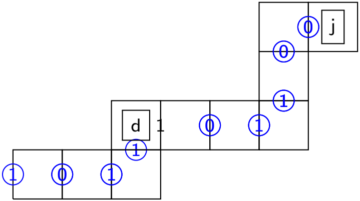


Theorem (Musiker, Schiffler, and Williams, 2009–10)

Each cluster variable (of a cluster algebra from a surface) can be written as the sum of the weights of all perfect matchings of a certain snake graph.
(Note: this demonstrates that the Laurent polynomial expansion has all positive coefficients).

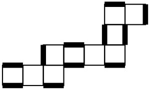
Bijection from subwords to perfect matchings (G.)

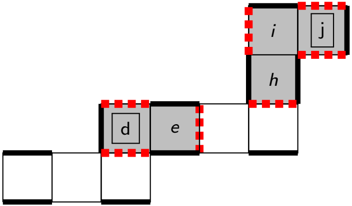
- Highlight the internal edges of G corresponding to s.
  Example: $w = 1011101100$ and subword $s = 101101100$.
  Highlight the edges $\underline{1}\ \underline{0}\ \underline{1}\ \underline{1}\ 1\ \underline{0}\ \underline{1}\ \underline{1}\ \underline{0}\ \underline{0}$ below.
- For each path L of consecutive highligted edges, let $\square_L$ be the tile which is north/east of the last edge in L.

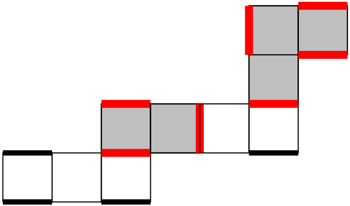


$\square_{1011} = \boxed{d}$ and $\square_{01100} = \boxed{j}$

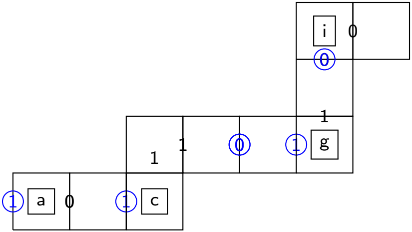
Bijection from subwords to perfect matchings (G.)

- Let $P_{\min}$ be the perfect matching which contains the first south edge and only boundary edges. 
- Let $fil(\square_L)$ be the minimal connected sequence of tiles such that $\square_L \in fil(\square_L)$ and the edges bounding $fil(\square_L)$ not in $P_{\min}$ forms a perfect matching of $fil(\square_L)$.
- Let $fil(s) = \bigcup_L fil(\square_L)$. Let $pm(s) = \{\text{edges bounding } fil(s)\} \ominus P_{\min}$.

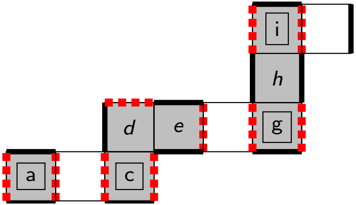


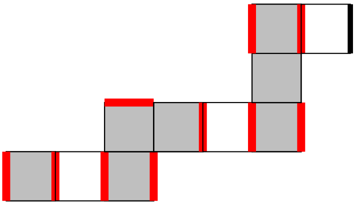


Another example: From 11010 to a perfect matching



$\square_{1} = \boxed{a}$, $\square_{1} = \boxed{c}$, $\square_{01} = \boxed{g}$, and $\square_{0} = \boxed{i}$.

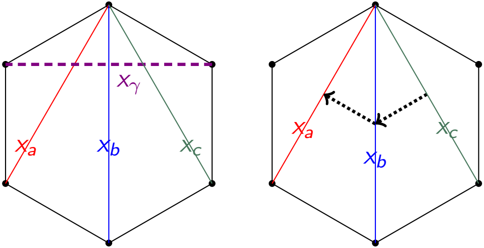




Thank you

Comments and suggestions are welcome

Example: arc to poset



Let $Q_\gamma = $
$$\begin{tikzcd} a \\ b \arrow[u] \\ c \arrow[u] \end{tikzcd}$$

Motivation:

Theorem (Musiker, Schiffler, and Williams, 2011)

The order filters of Q_γ are in bijection with the terms of the cluster variable expansion of x_γ with respect to x_a, x_b, and x_c (where the set of elements of each order filter corresponds to a term in the F-polynomial).