Isaac Newton Institute CAR Algebra Seminar
Tuesday, 28 September 2021

Talk notes at equnawan.github.io/talks/ini21.pdf

Cambrian combinatorics on quiver representations (type $\mathbb{A}$)

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(arXiv: 1912.02840)

Outline

1. $\eta : S_{n+1} \twoheadrightarrow \{\Delta\text{tions}\}$ $\underbrace{\text{Cambrian lattice}}_{\substack{\text{type } \mathbb{A} \text{ cluster algebra} \\ \text{exchange graph oriented} \\ \text{by green mutations}}}$

2. η polygon model for rep Q

3. What do Δtions correspond to? $\underbrace{\text{Maximal almost rigid}}_{\text{mar}}$ representations

4. The endomorphism algebra of a mar representation is a type $\mathbb{A}$ tilted algebra.

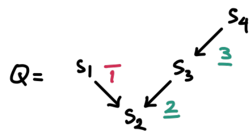
5. A stability function where each $M \in \operatorname{ind} Q$ is stable.

6. $\eta^{rep}: S_{n+1} \twoheadrightarrow \operatorname{mar}(Q)$ Cambrian lattice

1. Cambrian lattice from η (Reading 2004)

Q quiver of type A_{n+2}

$s_1 - s_2 - \cdots - s_{n+2}$

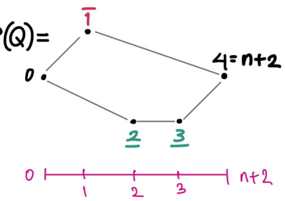


Type A_{2+2}

if $s_i \searrow s_{i+1}$ in Q, let $\bar{i}$ be up

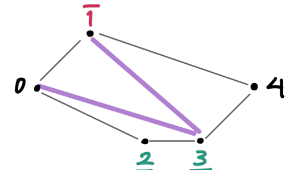
if $s_i \swarrow s_{i+1}$ in Q, let $\underline{i}$ be down

P(Q)=



$\eta_Q : \underbrace{S_{n+1}}_{\text{symmetric group}} \twoheadrightarrow \left\{ \begin{matrix} \text{triangulations} \\ \text{of } P(Q) \end{matrix} \right\}$

$w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3 \mapsto$



along top edges

paths from 0 to n+2

$\lambda_3 \quad 0 \ \bar{1} \ \underset{\text{remove } 3}{\times} \ 4 \qquad w(3) = \underline{3}$

$\lambda_2 \quad 0 \ \underset{\text{insert } \bar{1}}{\bar{1}} \ \underline{3} \ 4 \qquad w(2) = \bar{1}$

$\lambda_1 \quad 0 \ \underset{\text{remove } 2}{\times} \ \underline{3} \ 4 \qquad w(1) = \underline{2}$

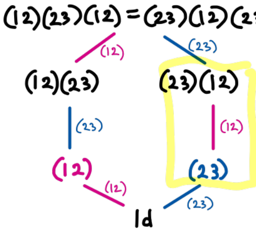
$\lambda_0 \quad 0 \ \underline{2} \ \underline{3} \ 4$

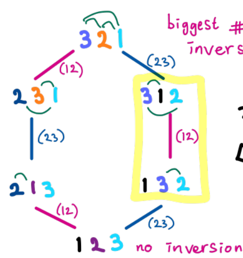
along bottom edges

The (right) weak order on S_{n+1} is a $\overbrace{\text{partial order (poset)}}^{\text{In fact, a lattice}}$ whose Hasse diagram is the Cayley graph of S_{n+1} with generators $\{(1,2), (2,3), \ldots, (n,n+1)\}$

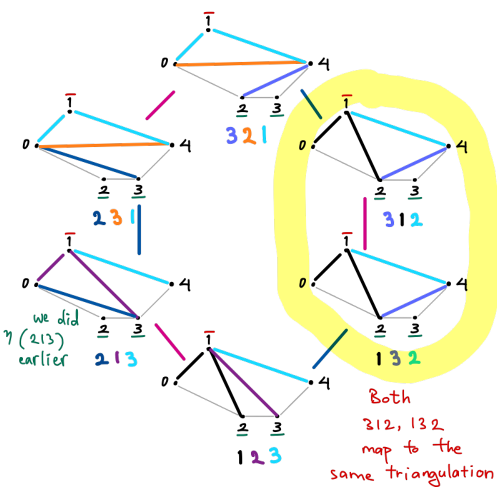
E.g. $(12)(23)(12) = (23)(12)(23)$

S_3





η_Q



$\eta_Q : S_{n+1} \xrightarrow[\text{from 0 to n+2}]{\text{union of paths}} \left\{ \begin{matrix} \text{triangulations} \\ \text{of } P(Q) \end{matrix} \right\}$

gives a quotient of the weak order called **Q-Cambrian lattice.**

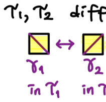
T_1 is covered by T_2

$T_1 \lessdot T_2$

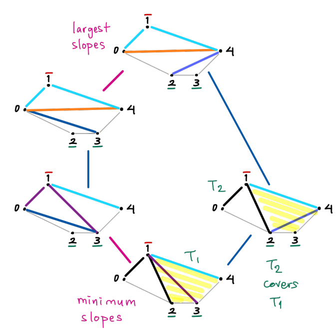
T_2
| covering relation
T_1

if:

- T_1, T_2 differ by a diagonal flip



- The diagonal γ_2 has larger slope



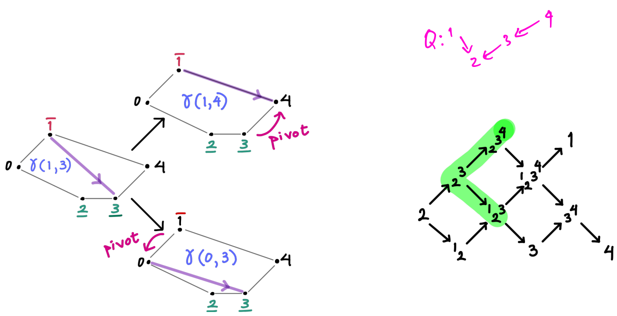
$\boxed{\text{2. } \eta \text{ polygon model for rep } Q}$

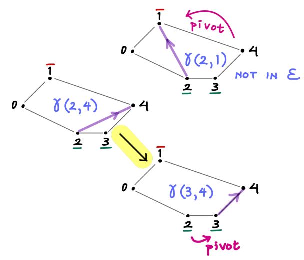
Special case of Opper–Plamondon–Schroll 2018 model for $\mathcal{D}^b(\text{mod } A)$
gentle

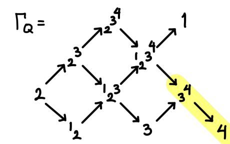
$\mathcal{E} := \{ \text{oriented line segments } \gamma(i,j) \mid 0 \le i < j \le n+2 \}$

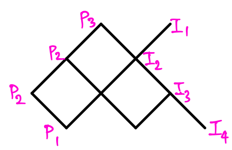
$\mathcal{C}_{P(Q)}$ category whose objects are $\mathcal{E}$

irreducible morphisms are pivoting one endpoint counterclockwise









<u>Thm A</u> There is an equivalence of categories

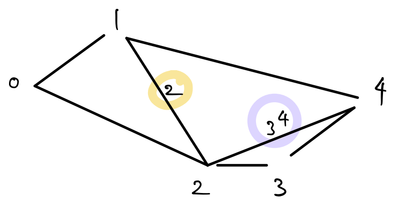
$$\mathcal{C}_{P(Q)} \longrightarrow \operatorname{Ind} Q$$

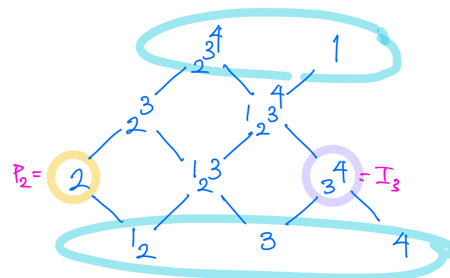
$$\gamma(i-1,j) \longleftrightarrow M(i,j) = \cdots 0 - \overset{i}{\Bbbk} \xrightarrow{1} \cdots \xrightarrow{1} \overset{j}{\Bbbk}$$

endpoint pivots $\longleftrightarrow$ irreducible morphisms

clockwise rotation $\longleftrightarrow$ AR translation τ

3. △tion corresponds to ... ?





<u>Def</u> T is <u>almost rigid</u> if

- T is basic (no repeated indecomposable summands)
- For each pair A, B of indecomposable summands of T,

  if $0 \to A \to E \to B \to 0$ is a non-split s.e.s

  then E is indecomposable.
- An almost rigid T is <u>maximal almost rigid</u> if

  $T \oplus M$ is not almost rigid for any representation M.

<u>Thm B</u> $\left\{ \begin{array}{c} \triangle\text{tions} \\ \text{of } P(Q) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{maximal almost} \\ \text{rigid representations of } Q \end{array} \right\}$ $=: \mathrm{mar}(Q)$

<u>Cor</u> Q type A_{n+2} quiver, $T \in \mathrm{mar}(Q)$

$\#\{\text{summands of } T\} = 2n+3$ (n+3 boundary line segments, n internal —"—)

$\#\,\mathrm{mar}(Q) = \frac{1}{n+2}\binom{2n+2}{n+1}$ Catalan numbers! :)

4. $\mathrm{End}_{\mathrm{rep}\,Q}\, T$ for $T \in \mathrm{mar}(Q)$

<u>Def</u> (Happel–Ringel 1982) Let M be a tilting module in rep Q.

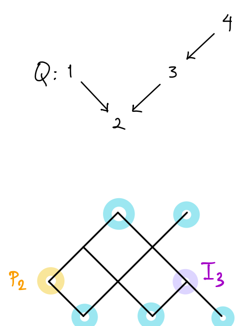
$\mathrm{End}_{\mathrm{rep}\,Q}$ is called a <u>tilted algebra of type Q</u>.

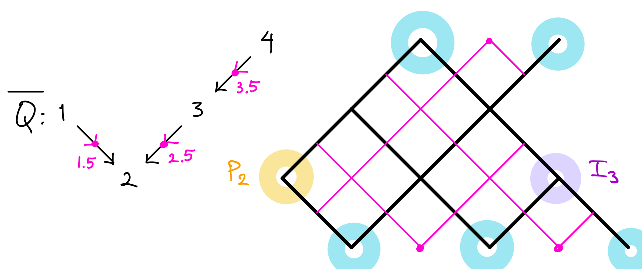
<u>Thm C</u> Let $T \in \mathrm{mar}(\overset{A_{n+2}}{Q})$, $C := \mathrm{End}_{\mathrm{rep}\,Q}(T)$.

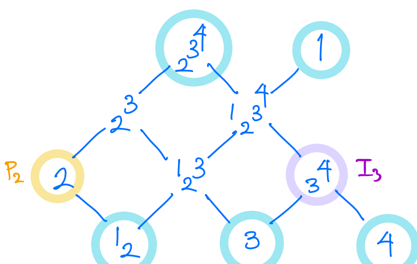
Then C is a tilted algebra of type $\overline{Q}$

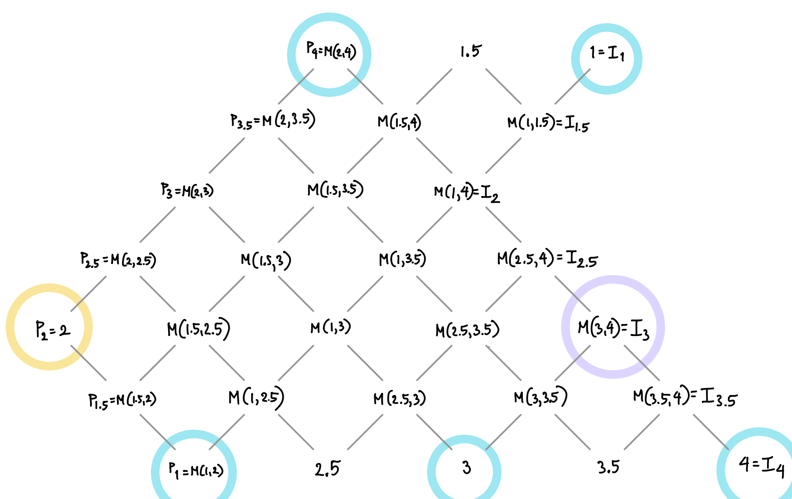
A_{2n+3}

$n+2+n+1$









summands of T are circled

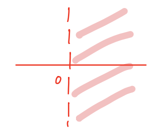
summands of $\overline{T}$ are circled

$$\underset{\text{mar}}{\mathrm{End}_{\mathrm{rep}\,Q}\, T} \cong \underset{\text{tilting in rep } \overline{Q}}{\mathrm{End}_{\mathrm{rep}\,\overline{Q}}\, \overline{T}}$$

5. A stability function where each $M \in$ ind Q is stable.

Let $K_0 = \mathbb{Z}^{\{\text{simples } s_x \mid x \in Q_0\}}$ be the Grothendieck group of rep Q.

- A <u>stability function</u> on rep Q is a group homomorphism



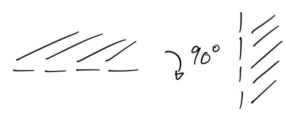
$$Z: K_0 \longrightarrow \mathbb{C}$$
$$[M] \longmapsto r(M)\, e^{i\pi\phi(M)} \qquad \text{s.t}$$

$\forall\ 0 \neq M$, $Z([M]) \in$ strict right half plane $\mathbb{H} := \{ re^{i\pi\phi} \mid r > 0, -\frac{1}{2} < \phi < \frac{1}{2} \}$

- $\phi(M)$ is called the <u>phase</u> of M

Note: Bridgeland uses strict upper half plane,

but it's convenient for us to rotate ↷ 90°



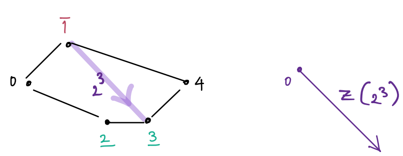
<u>Def</u> (Stability function from our polygon model)

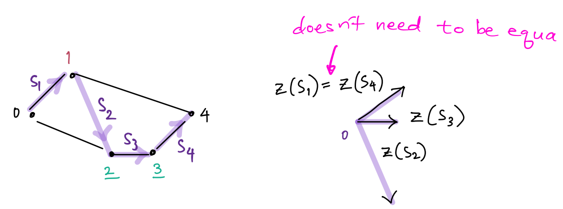
- Define $Z: K_0 \longrightarrow \mathbb{C}$ → vectors in $\mathbb{R}^2$

$$Z(\underbrace{[M(j,k)]}_{M:=}) = \text{vec}(\gamma(j-1,k))$$ vector w/ same direction & magnitude as $\gamma(j-1,k)$

$$= \underbrace{r(M)}_{\text{magnitude of } \gamma(j-1,k)}\ e^{i\,\overbrace{\arg(M)}}$$ → argument of $\text{vec}(\gamma(j-1,k))$, $-\frac{\pi}{2} < \arg(M) < \frac{\pi}{2}$

and extend additively to all of K_0.





- The phase of M is $\phi: \text{rep}\, Q \longrightarrow \mathbb{R}$
$$M \longmapsto \frac{1}{\pi}\arg(M) \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Thm D Every indecomposable M is stable with respect to our z.

(i.e. $\phi(L) < \phi(M)$ for all $0 \neq L \subsetneq M$)

proper subrepresentation

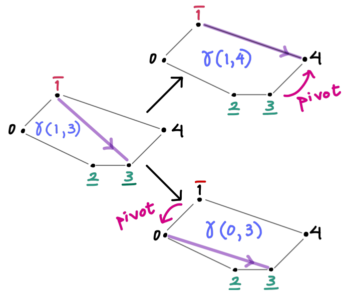
Pf If $0 \neq L$ is a proper subrepresentation of M,

the inclusion $L \hookrightarrow M$ is a nonzero morphism in $\mathrm{Hom}(L, M)$.

Since morphism $\gamma(L) \longrightarrow \gamma(M)$ is a sequence of counterclock pivots

like

each increases angle



so $\phi(L) = \frac{1}{\pi}\arg(L) < \frac{1}{\pi}\arg(M) = \phi(M)$ $\square$

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6. $\eta^{rep}: S_{n+1} \twoheadrightarrow \mathrm{mar}(Q)$ Cambrian lattice

Def poset on $\mathrm{mar}(Q)$

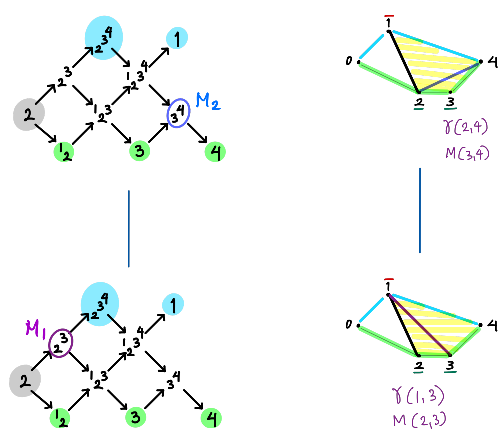
T_2 — covering relation — T_1 if :

(1) T_1, T_2 differ by one indecomposable summand $\underset{\text{in } T_1}{M_1} \sim \underset{\text{in } T_2}{M_2}$

(2) There is a short exact sequence

$0 \longrightarrow M_1 \longrightarrow A \oplus B \longrightarrow M_2 \longrightarrow 0$

where A, B are indecomposable summands of T_1 / M_1



Thm E The above poset on $\mathrm{mar}(Q)$ is a Cambrian lattice.

Rem The minimum mar module contains all projectives.

The maximum mar module contains all injectives.

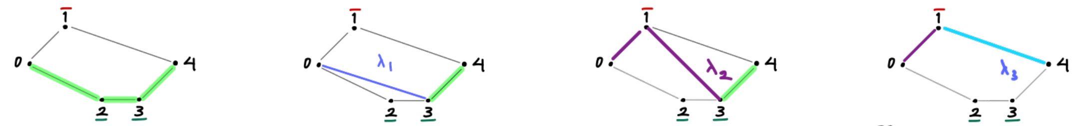
(Extra page)

$$\eta: S_{n+1} \xrightarrow{\text{union of paths}} \left\{\begin{matrix}\text{triangulations} \\ \text{of } P(Q)\end{matrix}\right\} \quad \text{induces} \quad \eta^r: S_{n+1} \xrightarrow[\underbrace{\text{with dimension vector}}_{\text{dim}} (1,1,\dots,1)]{\text{"union" of representations}} \text{mar}(Q)$$

E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\quad w(1) = \underline{2}$ $\quad w(2) = \overline{1}$ $\quad w(3) = \underline{3}$

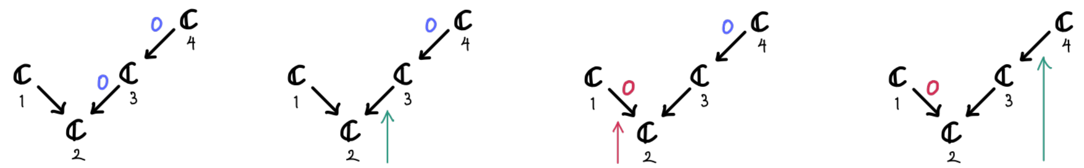
remove $\underline{2}$ $\quad$ insert $\overline{1}$ $\quad$ remove $\underline{3}$

$\lambda_0 \;\; 0\ \underline{2}\ \underline{3}\ 4$ $\quad \lambda_1 \;\; 0\ \times\ \underline{3}\ 4$ $\quad \lambda_2 \;\; 0\ \overline{1}\ \underline{3}\ 4$ $\quad \lambda_3 \;\; 0\ \overline{1}\ \times\ 4$



Take union of all edges of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to get triangulation $\eta(213)$

$\lambda_0^r = {}^1_2 \oplus 3 \oplus 4$ $\quad \lambda_1^r = {}^1{}_2{}^3 \oplus 4$ $\quad \lambda_2^r = 1 \oplus {}_2{}^3 \oplus 4$ $\quad \lambda_3^r = 1 \oplus {}_2{}^{3^4}$



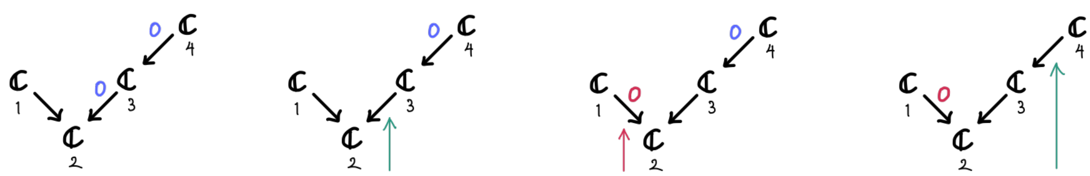
ext($\underline{2}$) $\quad$ deg($\overline{1}$) $\quad$ ext($\underline{3}$)

Take union of all 7 indecomposable summands to get m.a.r rep $\eta^r(213)$

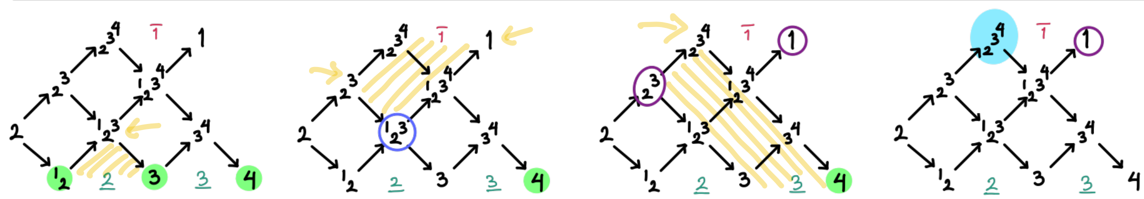
$$\eta^r: S_{n+1} \xrightarrow{\text{"union" of representations}} \text{mar}(Q) \quad \text{via "rectangles with missing corners"}$$

E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\quad w(1) = \underline{2}$ $\quad w(2) = \overline{1}$ $\quad w(3) = \underline{3}$

$\lambda_0^r = {}^1_2 \oplus 3 \oplus 4$ $\quad \lambda_1^r = {}^1{}_2{}^3 \oplus 4$ $\quad \lambda_2^r = 1 \oplus {}_2{}^3 \oplus 4$ $\quad \lambda_3^r = 1 \oplus {}_2{}^{3^4}$



ext($\underline{2}$) $\quad$ deg($\overline{1}$) $\quad$ ext($\underline{3}$)



Take union of all 7 circled indecomposables to get m.a.r rep $\eta^r(213)$