Cluster algebras and friezes

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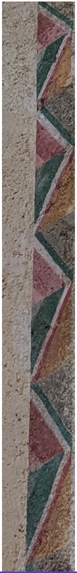
Frieze

In architecture, a *frieze* is an image that repeats itself along one direction.


Conway and Coxeter, 1970s

Definition

A **Conway – Coxeter frieze pattern** is an array of positive integers such that:

1. it is bounded above and below by a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the diamond rule $ad - bc = 1$.

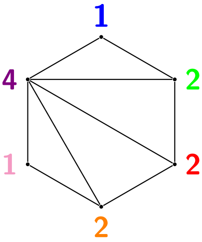
$$\begin{array}{cccccccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & \\
& 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & & & \\
& & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & & \\
& & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Conway and Coxeter, 1970s

Theorem

A Conway – Coxeter frieze pattern with n nontrivial rows $\longleftrightarrow$ *a triangulation of an* $(n+3)$*-gon.*

1		1		1		1		1		1		1		1		1		1		1		1				
	1		4		1		2		2		2		1		4		1		2		2		2			
		3		3		1		3		3		1		3		3		1		3		3		1		
			2		2		1		4		1		2		2		2		1		4		1		2	
				1		1		1		1		1		1		1		1		1		1		1		1



Fomin and Zelevinsky, 2001

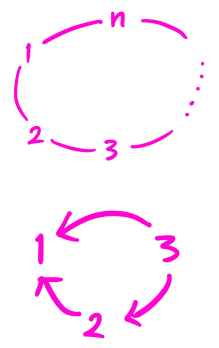
Start with a quiver (directed graph) Q on n vertices with no loops and no 2-cycles.

Example: type $\widetilde{\mathbb{A}}_{p,q}$

An acyclic quiver Q is of type $\widetilde{\mathbb{A}}_{p,q}$ if and only if

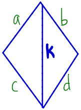
- its underlying graph is a circular graph with $n = p + q$ vertices,
- the quiver Q has p counterclockwise arrows and q clockwise arrows

For example, this is a quiver of type $\widetilde{\mathbb{A}}_{1,2}$ $\rightarrow$

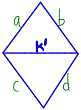


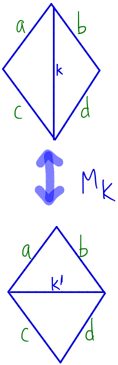
Annulus with p+q marked points on the boundary (Fomin-Shapiro-Thurston, 2006)

- An **arc** is an internal curve between marked points
- A **triangulation** is a maximal collection of non-crossing arcs
- A **flip** M_K replaces



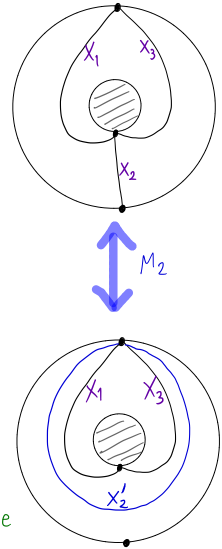
with



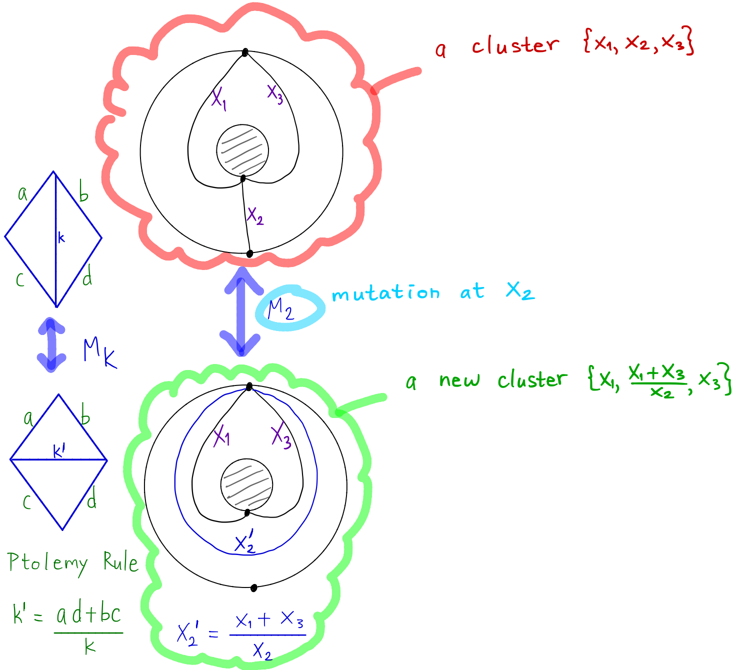


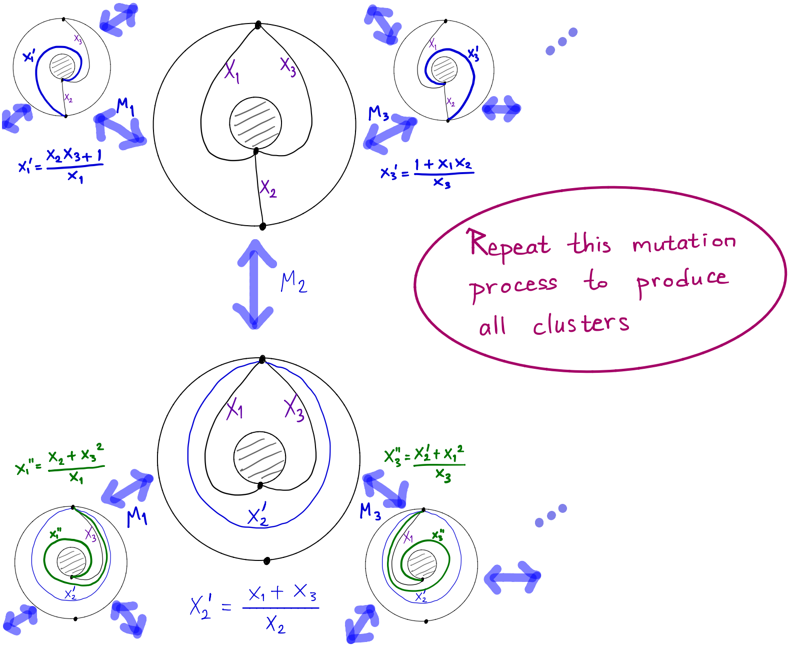
Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$

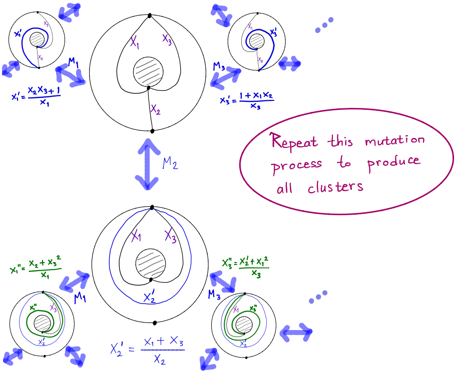


$$X_2' = \frac{X_1 + X_3}{X_2}$$


a cluster $\{x_1, x_2, x_3\}$
x_1
x_3
x_2
mutation at x_2
M_2
a new cluster $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$
x_1
x_3
x_2'
$x_2' = \frac{x_1 + x_3}{x_2}$
a
b
k
c
d
M_K
a
b
k'
c
d
Ptolemy Rule
$k' = \frac{ad+bc}{k}$



$x_1' = \frac{x_2 x_3 + 1}{x_1}$
M_1
X_1
X_3
X_2
M_3
$x_3' = \frac{1 + x_1 x_2}{x_3}$
M_2
Repeat this mutation process to produce all clusters
$x_1'' = \frac{x_2 + x_3^2}{x_1}$
X_2'
$x_3'' = \frac{x_2' + x_1^2}{x_3}$
$X_2' = \frac{X_1 + X_3}{X_2}$

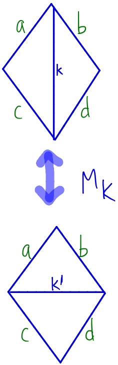


Def (Fomin – Zelevinsky, 2001)

- { cluster variables } =

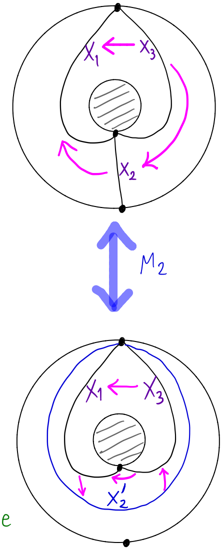
$$\bigcup_{\text{all clusters } \mathbf{x}} \{ \text{ elements of } \mathbf{x} \}$$

- The **cluster algebra** $\mathcal{A}(Q)$ is the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables.



Ptolemy Rule

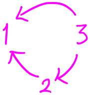
$$k' = \frac{ad+bc}{k}$$

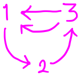


$$X_2' = \frac{X_1 + X_3}{X_2}$$

If j follows i counterclockwise along a triangle

then draw $j \leftarrow i$





Friezes

Definition

Let Q be a quiver and $\mathcal{A}(Q)$ the cluster algebra from Q. A **frieze** of type Q is a ring homomorphism $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Z}$ which maps every cluster variable to a positive integer.

Example:

- A frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Z}$ defined by fixing a cluster $\mathbf{x}$ and sending each cluster variable in $\mathbf{x}$ to 1.

Friezes examples

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & \frac{x_2+1}{x_1} & & x_1 \\
x_2 & & \frac{x_1x_3+1}{x_2} & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & x_2 & \\
 & x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & \frac{x_2+1}{x_3} & & x_3 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Figure: The cluster variables of the cluster algebra $\mathcal{A}(Q)$ for the type $\mathbb{A}_3$ quiver $Q = 1 \to 2 \leftarrow 3$.

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & 1 & & 3 & & 2 & & 1 \\
1 & & 2 & & 5 & & 1 & \\
 & 1 & & 3 & & 2 & & 1 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Figure: Setting $x_1{=}x_2{=}x_3{=}1$ produces a Conway – Coxeter frieze pattern.

Unitary friezes

Definition

We say that a frieze $\mathcal{F}$ is **unitary** if there exists a cluster $\mathbf{x}$ in $\mathcal{A}(Q)$ such that $\mathcal{F}$ maps every cluster variable in $\mathbf{x}$ to 1.

Remark

All friezes of type $\mathbb{A}$ are unitary (due to Conway and Coxeter), but there are non-unitary friezes of type $\mathbb{D}$, $\widetilde{\mathbb{D}}$, $\mathbb{E}$, and $\widetilde{\mathbb{E}}$ (due to Fontaine and Plamondon).

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Theorem 2 (G – Schiffler)

All friezes of type $\widetilde{\mathbb{A}}_{p,q}$ are unitary.

Example: There are the two friezes of type $\widetilde{\mathbb{A}}_{1,2}$, up to translation.

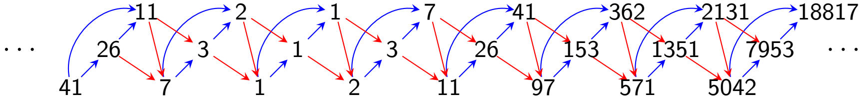


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

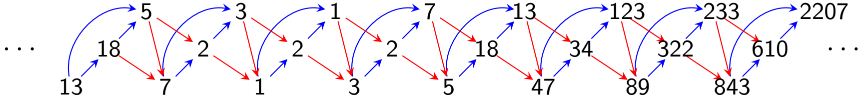
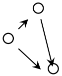
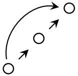


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Every acyclic shape, for example,  and  tells us the frieze values of a cluster.

Example (A possible step in the algorithm)

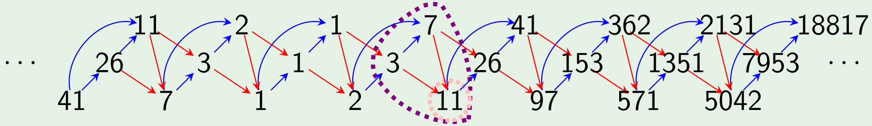
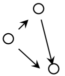
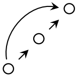


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

Mutating at the position with frieze value 11 produces a new frieze value $\frac{3\times 7+1}{11} = 2 < 11$.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Every acyclic shape, for example,  and  tells us the frieze values of a cluster.

Example (A possible step in the algorithm)

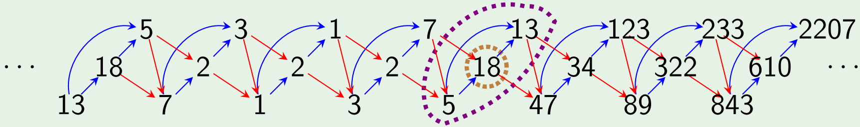


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

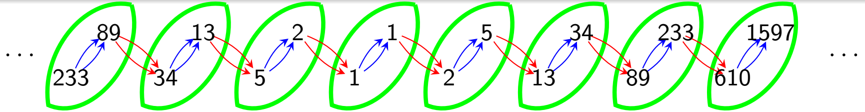
Mutating at the position with frieze value 18 (which is not a sink/source) produces a new frieze value $\frac{5+13}{18} = 1$.

Frieze vectors

Definition

Fix a cluster $\mathbf{x} = (x_1, \ldots, x_n)$.

- A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ can be used to define a homomorphism $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Q}$ by defining $\mathcal{F}(x_i) = a_i$ for all $i = 1, \ldots, n$.
- We say that $(a_1, \ldots, a_n)$ is a **frieze vector** relative to $\mathbf{x}$ if $\mathcal{F}$ maps every cluster variable to a positive integer.
- If $(a_1, \ldots, a_n)$ determines a unitary frieze, we say that $(a_1, \ldots, a_n)$ is a **unitary** frieze vector.



The slices display the frieze vectors

$\ldots, (233, 89), (34, 13), (5, 2), (1, 1), (2, 5), (13, 34), (89, 233), (610, 1597), \ldots$

relative to a cluster with the quiver $1 \Rightarrow 2$.

Frieze vectors algorithm

Proposition 3

A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is a frieze vector relative to an acyclic Q iff a_k divides

$$\prod_{k \to j \text{ in } Q} x_j + \prod_{k \leftarrow j \text{ in } Q} x_j$$

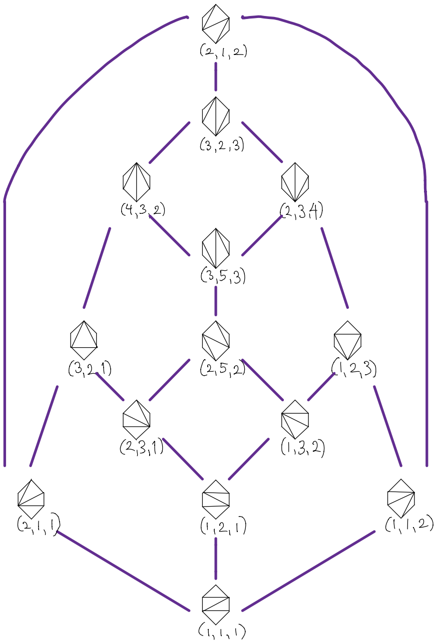
for all $k = 1, \ldots, n$.

Example

A vector $(a_1, a_2, a_3) \in \mathbb{Z}^3_{>0}$ is a positive frieze vector relative to $1 \to 2 \leftarrow 3$ iff

$$\frac{a_2 + 1}{a_1}, \frac{a_1 a_3 + 1}{a_2}, \frac{a_2 + 1}{a_3}$$

are integers.



Frieze vectors

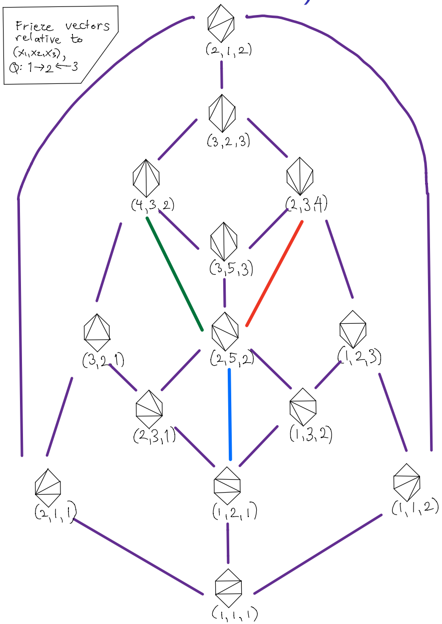
Theorem 4 (G – Schiffler)

Fix $\mathcal{A}(Q)$ and fix an arbitrary cluster $\mathbf{x} = (x_1, \ldots, x_n)$. Then there is a bijection between clusters in $\mathcal{A}(Q)$ and unitary frieze vectors relative to $\mathbf{x}$.

Friezahedron (further questions, with Castillo)

In type $\mathbb{A}_n$, $\mathbb{D}_n$, and $\mathbb{E}_6$, it is known that there are finitely many positive integral frieze vectors. Take the convex hull of these points in $\mathbb{R}^n$.

```
sage: V = [ [1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 1], [2, 1, 2], [2, 3, 1], [2, 3, 4], [2, 5, 2], [3, 2, 1], [3, 2, 3], [3, 5, 3], [4, 3, 2] ]
sage: P = Polyhedron(V)
```



Finite-type friezes

Classify quivers with finitely many friezes (up to cluster automorphism).

- Quivers that have finitely many friezes (up to cluster autormosphism): $\mathbb{A}, \widetilde{\mathbb{A}}_{p,q}, \mathbb{D}$, and $\mathbb{E}_6$.
- Quivers that I think may have finitely many friezes (up to cluster automorphism): rank 2, $\mathbb{E}_6$, $\mathbb{E}_7$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, and quivers with triangulation model.

References

Emily Barnard, Emily Gunawan, Emily Meehan, and Ralf Schiffler.
Cambrian combinatorics on quiver representations (type A), 2019.
Preprint, 1912.02840.

J. H. Conway and H. S. M. Coxeter.
Triangulated polygons and frieze patterns.
Math. Gaz., 57:87–94, 175–183, 1973.

B. Fontaine and P.-G. Plamondon.
Counting friezes in type D_n.
J. Algebraic Combin., 44(2):433–445, 2016.

S. Fomin, M. Shapiro, and D. Thurston.
Cluster algebras and triangulated surfaces. I. Cluster complexes.
Acta Math., 201(1):83–146, 2008.

S. Fomin and A. Zelevinsky.
Cluster algebras. I. Foundations.
J. Amer. Math. Soc., 15(2):497–529, 2002.

E. Gunawan and R. Schiffler.
Frieze vectors and unitary friezes.
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Comments and questions

Thank you!