Combinatorial and algebraic interpretation of the Catalan numbers

Intro: Catalan numbers, 200+ Catalan objects

1 Conway – Coxeter friezes

Row of 0's
Row of 1's
Pos integers

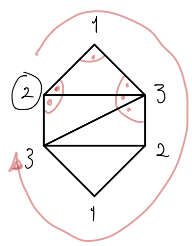
Rule

$$\det \begin{pmatrix} & b & \\ a & & d \\ & c & \end{pmatrix} = ad - bc = 1$$

$$\begin{array}{ccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & \\
2 & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & & 1 & & 3 & & \cdots \\
 & 1 & & 2 & & 5 & & 1 & & 2 & & 5 & & 1 & & 2 & & 5 & \\
 & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & & 1 & & 3 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & & & & 0 & & 0 & & 0 & & & & & & & & & &
\end{array}$$

rank $n = 3$

might be fun for teaching kids arithmetic



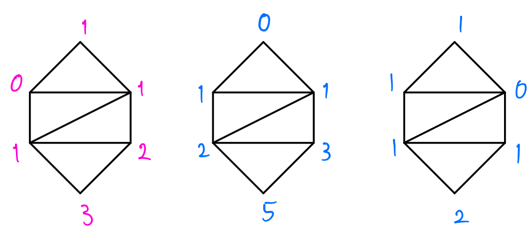
Thm (Conway, Coxeter, 1970s)

$$\left\{ \begin{matrix} \text{[ConCox] friezes} \\ \text{rank } n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{triangulations} \\ \text{of } (n+3)\text{-gon} \end{matrix} \right\}$$

Rem Both are Catalan objects, counted by C_{n+1}

where $C_n = \frac{1}{n+1}\binom{2n}{n}$, $\underbrace{\binom{2n}{n}}_{\text{a binomial coefficient}} := \frac{(2n)!}{n!\,n!}$

Primary School Algorithm



"Replace integers with"

Rational functions in x, y, z

$\boxed{?}\, y - \frac{(y+1)^2}{xz} = 1$

$$\boxed{?} = \frac{1 + \frac{(y+1)^2}{xz}}{y} = \frac{xz + y^2 + 2y + 1}{xyz}$$

$$\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\frac{y+1}{z} & & z & & \frac{xz+1+y}{yz} & & \frac{y+1}{x} & & x & & \\
 & y & & \frac{xz+1}{y} & & \boxed{?} & & y & & \frac{xz+1}{y} & \\
 & & x & & \frac{xz+1+y}{xy} & & \frac{y+1}{z} & & z & & \\
 & & & 1 & & 1 & & 1 & & 1 & \\
 & & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

- get 9 cluster variables
- Note: all are Laurent polys in x, y, z

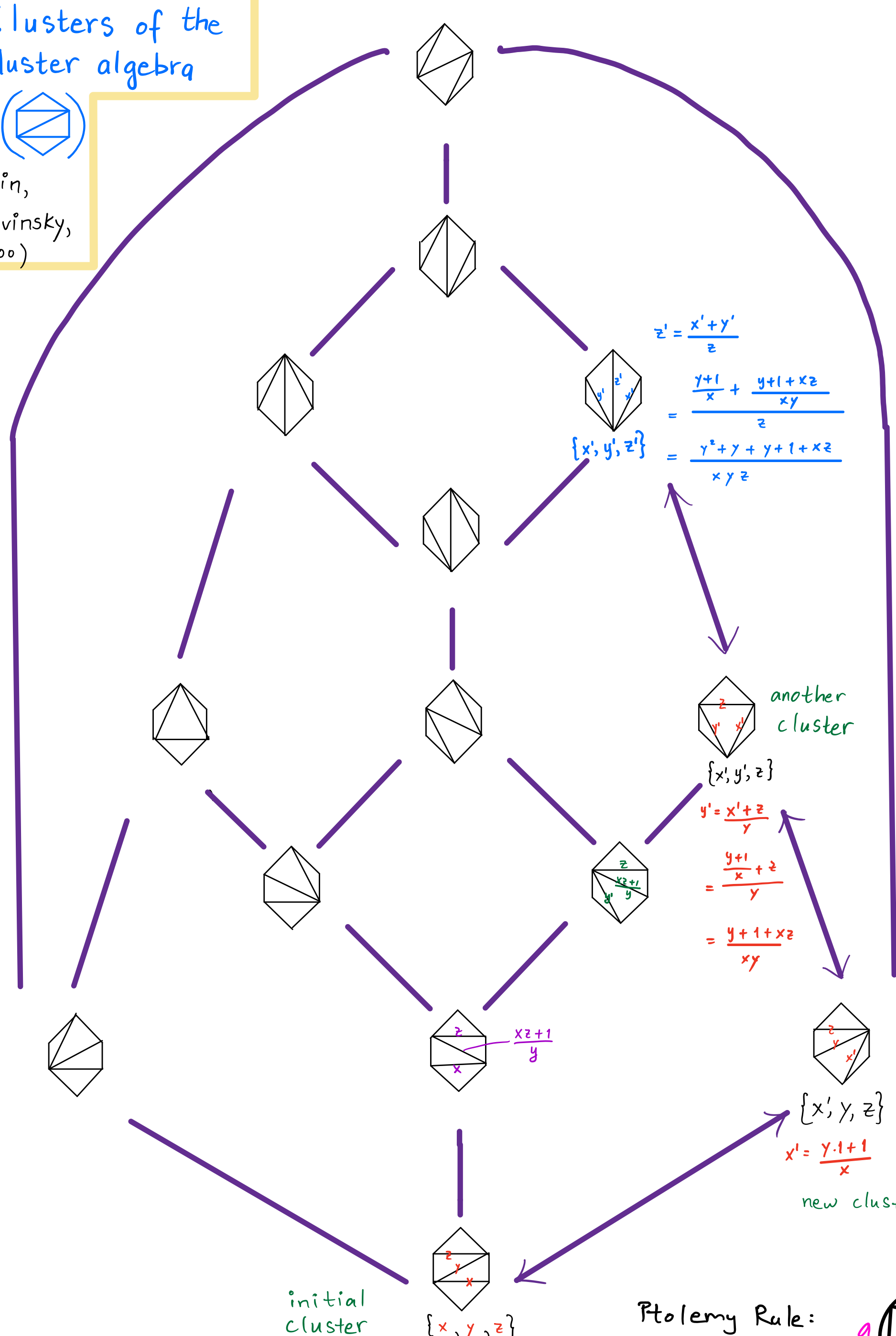
$$\Downarrow \quad \frac{\text{polynomial}}{\text{monomial}}$$

(Not obvious)

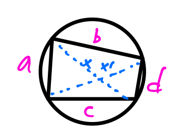
2. Clusters of the cluster algebra

$\mathcal{A}$( )

(Fomin, Zelevinsky, 2000)

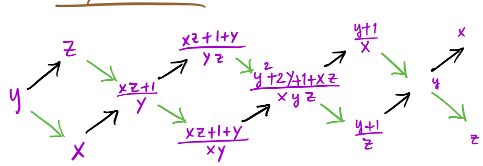


Ptolemy Rule:



$$xx' = ad + bc$$

my note:



$n=3$

Repeat this mutation process to produce all clusters ($C_4 = 14$ clusters)

$\{\text{cluster variables}\} = \bigcup_{\text{clusters}} x$. $\left(\sum_{i=2}^{n+1} i = 9 \text{ cluster variables}\right)$

The cluster algebra $\mathcal{A}(Q)$ is the subring of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all cluster variables.

The cluster algebra $\mathcal{A}(\;)$ is the "subring" of $\mathbb{Q}(x,y,z) = \left\{\begin{matrix}\text{rational} \\ \text{functions} \\ \text{in } x,y,z\end{matrix}\right\}$ "generated" by all cluster cluster variables, meaning

$\mathcal{A} = \{\text{polynomials in the cluster variables}\}$

e.g. $-5x^2y + 100\left(\frac{y+1}{x}\right)z' \in \mathcal{A}$

Thm

$\left\{\begin{matrix}\text{clusters of} \\ \mathcal{A}(\;)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{triangulations} \\ \text{of } (n+3)\text{-gon}\end{matrix}\right\}$

"Hence the clusters are Catalan objects"

(Show η map on handout iPad)

3. Cambrian Combinatorics (Reading, Reiner)

Dynkin diagram A_r

Def A quiver (directed graph) of type A is an orientation of $\underset{1}{\circ}\!-\!\circ\!-\!\circ \cdots \circ\!-\!\underset{r}{\circ}$

Q is type A_{n+2} $P(Q)$ is $(n+3)$-gon

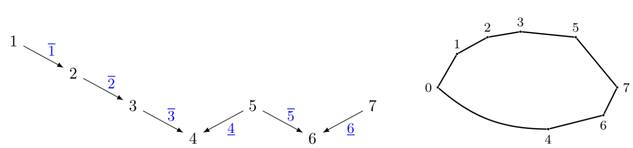


FIGURE 1. The polygon $P(Q)$ and quiver Q with $\underline{[6]} = \{4, 6\}$ and $\overline{[6]} = \{1, 2, 3, 5\}$.

Surjective map

$\eta : S_{n+1} \longrightarrow \left\{\begin{matrix}\text{triangulations} \\ \text{of } P(Q)\end{matrix}\right\}$

eta

"Give me a permutation"

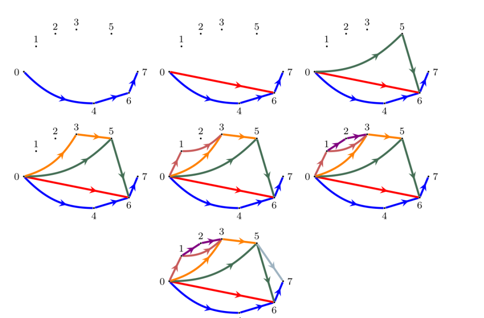


FIGURE 2. The paths $\lambda_i(453126)$ and triangulation for $\eta_Q(453126)$ from Example 2.3

4. Quiver Representations rep Q

Quiver:

$Q = 1 \rightarrow 2 \leftarrow 3$

quiver rep of Q: assignment of a vector space to each vertex, & linear map to each arrow

$V_1 \xrightarrow{\alpha} V_2 \xleftarrow{\beta} V_3$

E.g. $\mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{0} 0$ $\in \operatorname{Ind} Q$

$\mathbb{C}^2 \xrightarrow{\begin{bmatrix}1&1\\0&1\end{bmatrix}} \mathbb{C}^2 \xleftarrow{\begin{bmatrix}1\\1\end{bmatrix}} \mathbb{C} \cong \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C} \oplus \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{0} 0$

not in Ind Q

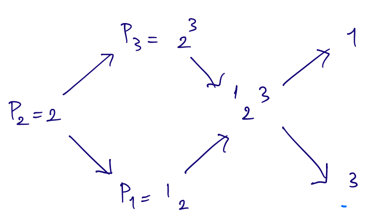
Def (Ind Q) $M \in \operatorname{rep} Q$ is indecomposable if it's not a direct sum of two nonzero representations.

Shorthand $\,{}^{2}{}_{3}{}_{4}{}^{5}$ means $0 \rightarrow V_2 = \mathbb{C} \rightarrow V_3 = \mathbb{C} \rightarrow V_4 = \mathbb{C} \leftarrow V_5 = \mathbb{C} \rightarrow 0$

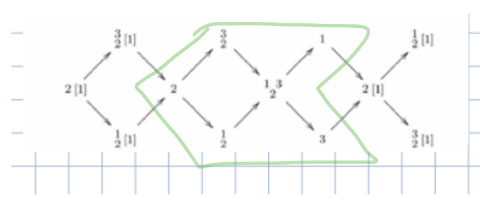
Gabriel: If Q is type A_n, Ind $Q \longleftrightarrow$ intervals of $\{1, \dots, n\}$

Auslander–Reiten quiver: vertices $\longleftrightarrow$ Ind Q; arrows $\longleftrightarrow$ irreducible morphism "does not factor nontrivially through another representation"

paths starting at 3



stop

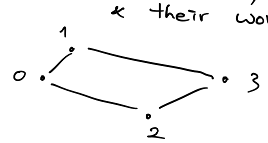


5. Geometric model for rep Q (Barnard, G, Meehan, Schiffler)

inspired by Reading & Reiner & their work on Cambrian lattices

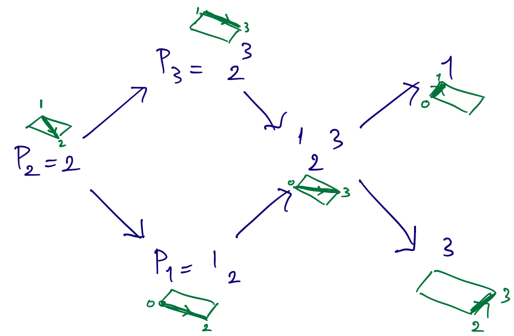
Show handout

$Q = 1 \xrightarrow{\bar{1}} 2 \xleftarrow{\bar{2}} 3$

$P(Q) =$ 

$\boxed{\text{Thm}}$ $\left\{\begin{array}{c}\text{line segments} \\ \text{in } P(Q)\end{array}\right\} \xleftrightarrow{\ F\ }$ Ind Q

line $i \longrightarrow j$, $i < j$ $\longmapsto$ $M(i+1, j)$ supported on vertices $i+1$ to j



pivots in $P(Q)$, i.e. moving one end pt counterclockwise $\longleftrightarrow$ irreducible morphisms

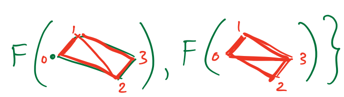
$\{\text{triangulations in } P(Q)\} \xleftrightarrow{\ F\ } \left\{\begin{array}{l}\text{what (Catalan)} \\ \text{objects ?}\end{array}\right\}$

Ans: maximal almost rigid representations $\text{mar}(Q)$.

Def (type A) • Say $T \cong M_1 \oplus \cdots \oplus M_t$ (no repeats) $\in$ rep Q is almost rigid if $\forall$ indecomposable summands A, B of T, either $\text{Ext}(A,B) = 0$ or there is a nontrivial short exact sequence

$$0 \longrightarrow B \to \underbrace{E}_{\text{indecomposable}} \longrightarrow A \longrightarrow 0$$

• Say T is maximal almost rigid if T is a.r, and $T \oplus M$ is not a.r $\forall$ $M \in$ rep Q.

Ex $\text{mar}\left(\begin{smallmatrix} 1 & & 3 \\ & \searrow\ 2 \swarrow & \end{smallmatrix}\right) = \{$  $\}$

$\boxed{\text{6. Further Questions}}$

- Modify $\text{mar}(Q)$ def & combinatorics for type D_n, $\tilde{A}_{p,q}$ (with Barnard, Meehan, Sch)
- Classify Q with finitely many friezes, up to cl. automorphism

Guess: $\underbrace{A, \tilde{A}_{p,q}, D, E_6}_{\text{"frieze finite type"}}$ $\underbrace{E_7, E_8, \tilde{D}, \tilde{E}, \text{quivers w/ triangulation model}}_{\text{still open}}$