Conway – Coxeter friezes, cluster algebras, and SageMath

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Sage Days 95: Women In Sage
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Outline

1. Conway – Coxeter friezes
    - A Conway – Coxeter frieze is a Catalan object
    - Connection to classical objects like continued fractions and binary words
    - Connection to quiver representations and cluster algebras
    - Using SageMath to draw friezes using LaTeX
2. Cluster algebras
    - Commutative algebras with a lot of combinatorial structure
    - Using SageMath to do cluster algebra computation

Frieze

A *frieze* is an image that repeats itself along one direction. The name comes from architecture, where a frieze is a decoration running horizontally below a ceiling or roof.

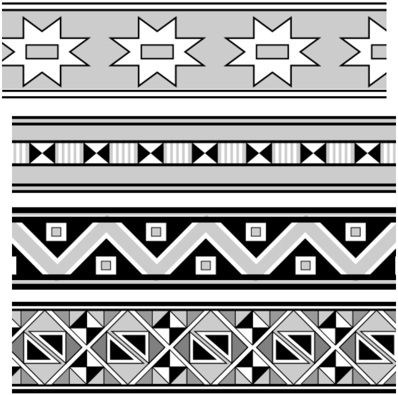


Figure: M. Ascher, *Ethnomathematics*, p162.

Conway-Coxeter frieze

Definition

A (type A) **frieze** is an array such that:

1. it is bounded above and below by a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

   satisfies the diamond rule $ad - bc = 1$.

A **Conway-Coxeter frieze** consists of only positive integers.

Example (an integer frieze)

$$\begin{array}{lcccccccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\text{Row 2} & \cdots & & \color{blue}{\mathbf{3}} & & \color{red}{\mathbf{1}} & & \color{lime}{\mathbf{2}} & & \color{purple}{\mathbf{2}} & & \color{magenta}{\mathbf{1}} & & 3 & & 1 & \\
 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
 & \cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Glide symmetry

A *glide symmetry* is a combination of a translation and a reflection.

		1		**1**		**1**		**1**		**1**		**1**		**1**			⋯
Row 2	⋯		**3**		**1**		**2**		**2**		**1**		**3**		**1**		
		2		**2**		**1**		**3**		**1**		**2**		**2**			⋯
	⋯		**1**		**1**		**1**		**1**		**1**		**1**		**1**		

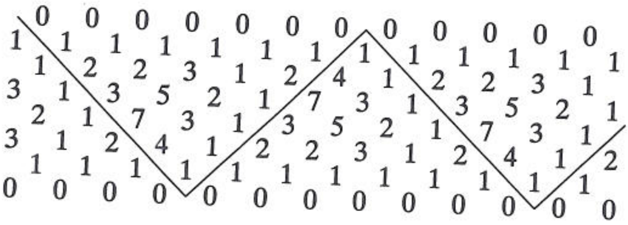


Table 1.

Children practicing arithmetic

Note: every frieze is completely determined by the 2nd row.

Frieze 1

0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1

1 2 2 3 1 2 4 1 2 2 3 1

3 5

7

Rule

Every

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies

$$ad - bc = 1$$

Frieze 2

0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

2 2 2 3 1 2 5 1 2 2 2 3 1

1 3 3 9

4

Children practicing arithmetic: Answer Key

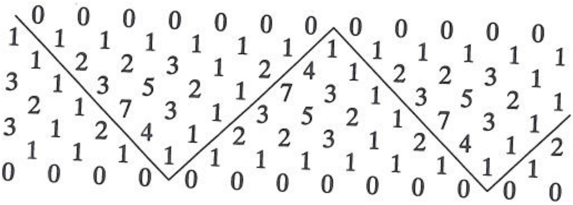


Table 1.

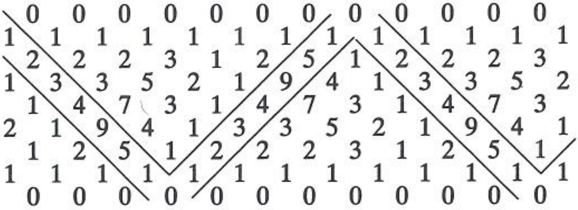
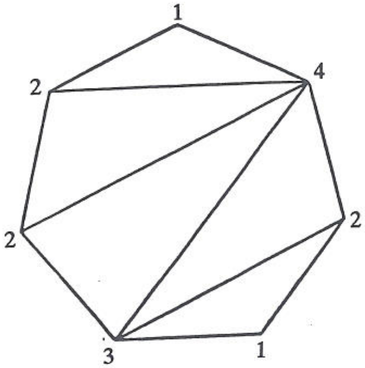
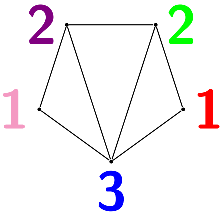


Table 2.

Frieze Sage demo

What do the numbers around each polygon count?





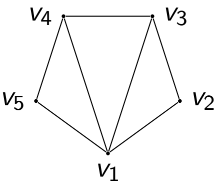
Conway and Coxeter (1970s)

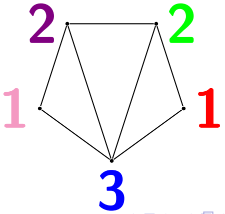
Theorem

A Conway – Coxeter frieze with n nontrivial rows $\longleftrightarrow$ *a triangulation of an* $(n+3)$*-gon*

Note: Hence Conway – Coxeter friezes are Catalan objects.

		1		1		1		1		1		1		1		$\cdots$
Row 2	$\cdots$		**3**		**1**		**2**		**2**		**1**		3		1	
		2		2		1		3		1		2		2		$\cdots$
	$\cdots$		1		1		1		1		1		1		1	





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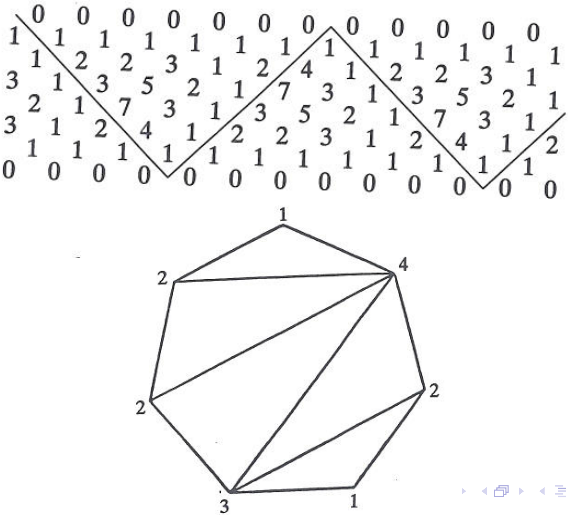


Primary school algorithm

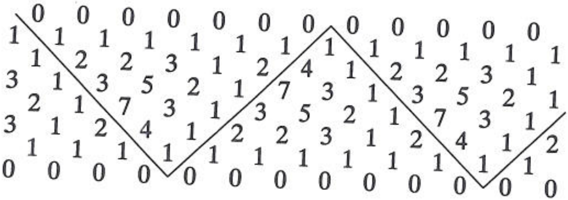
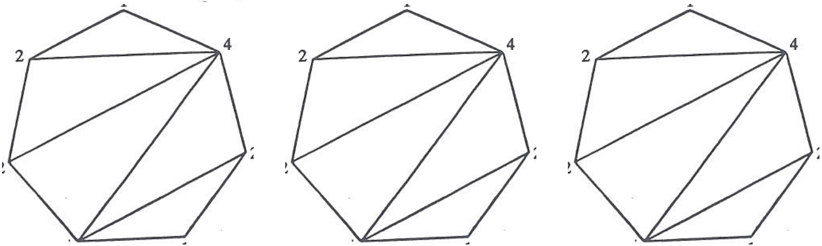


Table 1.

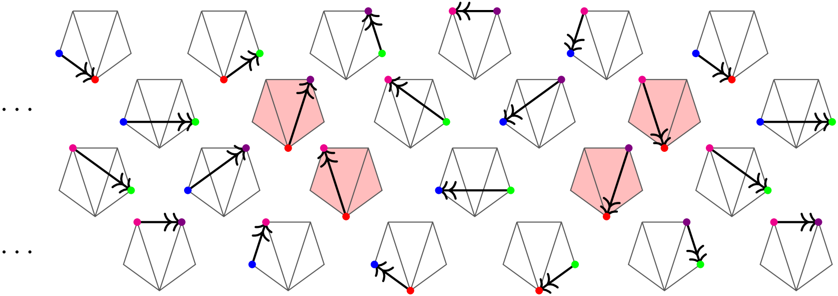


Broline, Crowe, and Isaacs (BCI, 1970s)

Theorem

Entries of a frieze $\longleftrightarrow$ edges between two vertices.

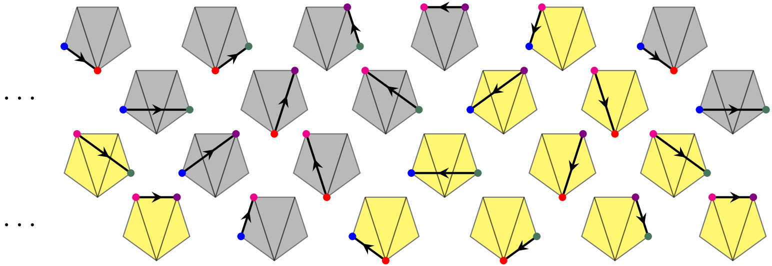
		1		1		1		1		1		1		$\cdots$
Row 2	$\cdots$		3		**1**		2		2		**1**		3	
		2		2		**1**		3		**1**		2		$\cdots$
	$\cdots$		1		1		1		1		1		1	



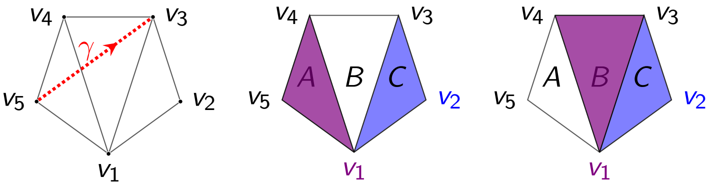
Glide symmetry (again)

Recall: A *glide symmetry* is a combination of a translation and a reflection. If we forget the arrows' orientation, the diagonals have glide symmetry.

		1		**1**		**1**		**1**		**1**		**1**		**1**		⋯
Row 2	⋯		**3**		**1**		**2**		**2**		**1**		**3**		**1**	
		2		**2**		**1**		**3**		**1**		**2**		**2**		⋯
	⋯		**1**		**1**		**1**		**1**		**1**		**1**		**1**	



Broline, Crowe, and Isaacs (BCI, 1970s)



T and γ $\quad (t_1, t_2) = (A, C) \quad (t_1, t_2) = (B, C)$

Definition (BCI tuple)

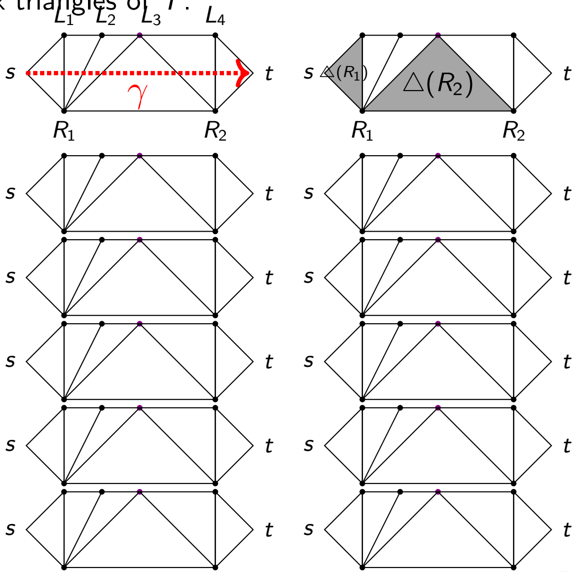
Let R_1, R_2, $\ldots$, R_r be the boundary vertices to the right of γ. A **BCI tuple** for γ is an r-tuple $(t_1, \ldots, t_r)$ such that:

(B1) the i-th entry t_i is a triangle of T having R_i as a vertex.

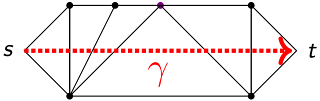
(B2) the entries are pairwise distinct.

How many BCI tuples are there?

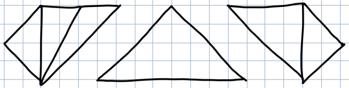
Example: A triangulation T of an octagon and a diagonal γ which crosses six triangles of T.



Continued fractions (Çanakçı, Schiffler)



Continued Fraction



3 , 1 , 2

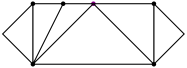
$$3 + \cfrac{1}{\left(1 + \cfrac{1}{2}\right)} \stackrel{\text{compute}}{=}$$

Binary number representations

A **binary number representation** is an expression of a nonzero integer in the base-2 numeral system, with digits consisting of 1s and $0s$. For example,

- $1 = 1 * 2^0$ (in decimal) is written as 1 (in binary).
- $2 = 1 * 2^1$ (in decimal) is written as 10 (in binary).
- $4 = 1 * 2^2$ (in decimal) is written as 100 (in binary).
- $5 = 1 * 2^2 + 1 * 2^0$ (in decimal) is written as 101 (in binary).
- $29 = 16 + 8 + 4 + 1 = 1 * 2^4 + 1 * 2^3 + 1 * 2^2 + 1 * 2^0$ is written as 11101 (in binary).

Subwords of binary numbers



We can think of "11101" as a word in the alphabets $\{0, 1\}$. The following are all the (scattered, non-consecutive) subwords of "11101" which start with "1":

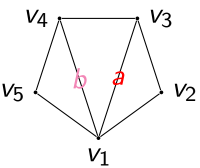
- 11101 (empty)
- $\underline{1}1101$: $\underline{1}$
- $\underline{1}11\underline{0}1$: $\underline{10}$
- $\underline{11}101$: $\underline{11}$
- $\underline{1}11\underline{01}$: $\underline{101}$
- $\underline{11}1\underline{0}1$: $\underline{110}$
- $\underline{111}01$: $\underline{111}$
- $\underline{11}1\underline{01}$: $\underline{1101}$
- $\underline{1110}1$: $\underline{1110}$
- $\underline{111}0\underline{1}$: $\underline{1111}$
- $\underline{11101}$: $\underline{11101}$ (the word itself)

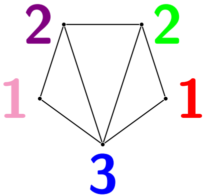
Cluster algebras (Fomin and Zelevinsky, 2000)

A **cluster algebra** is a commutative ring generated by sets called **clusters**. Each element of a cluster is called **cluster variables**.
Theorem (Caldero – Chapoton, 2006): The cluster variables of a cluster algebra from a triangulated polygon (type A) form a frieze.

$$
\begin{array}{ccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & \\
\cdots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

(Example: type A_2)





- Remark: If the variables are specialized to 1, we recover the Conway – Coxeter positive integer frieze.

Example: a type A_3 frieze

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & \frac{x_2+1}{x_1} & & x_1 \\
x_2 & & \frac{x_1x_3+1}{x_2} & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & x_2 & \\
 & x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & \frac{x_2+1}{x_3} & & x_3 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Frieze over the positive integers

Specializing $x_1 = x_2 = x_3 = 1$ gives a Conway – Coxeter positive integer frieze

1		1		1		1	
	1		3		2		1
1		2		5		1	
	1		3		2		1
1		1		1		1	

Frieze over the integers

Specializing $x_1 = x_2 = 1$ and $x_3 = -1$ gives

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & -1 & & -1 & & 2 & & 1 \\
1 & & 0 & & -3 & & 1 & \\
 & 1 & & 1 & & -2 & & -1 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Frieze over the Gaussian integers $\mathbb{Z}[i]$

Specializing $x_1 = 1$, $x_2 = i$, and $x_3 = i$ gives

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & i & & -1-2i & & 1+i & & 1 \\
i & & 1-i & & -3i & & i & \\
 & 1 & & 2-i & & 1-i & & i \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Frieze over the quadratic integer ring $\mathbb{Z}[\sqrt{-3}]$

Specializing $x_1 = 1$, $x_2 = \frac{1+\sqrt{-3}}{2}$, $x_3 = 1$ gives

$$
\begin{array}{cccccc}
1 & & 1 & & 1 & \\
& 1 & & 2-\sqrt{-3} & & \frac{3+\sqrt{-3}}{2} \\
\frac{1+\sqrt{-3}}{2} & & 1-\sqrt{-3} & & \frac{7-\sqrt{-3}}{2} & \\
& 1 & & 2-\sqrt{-3} & & \frac{3+\sqrt{-3}}{2} \\
1 & & 1 & & 1 &
\end{array}
$$

Cluster algebra Sage demo

Website references:

1. **Wikipedia entry**
   https://en.wikipedia.org/wiki/Cluster_algebra
2. Cluster Algebras Portal
   http://www.math.lsa.umich.edu/~fomin/cluster.html

arXiv.org references:

1. Introductory cluster algebra survey by Lauren Williams titled **Cluster algebras: an introduction**
   https://arxiv.org/abs/1212.6263
2. Cluster algebra textbook by Sergey Fomin, Lauren Williams, Andrei Zelevinsky titled **Introduction to cluster algebras**
   https://arxiv.org/abs/1608.05735
3. Frieze survey by Sophie Morier-Genoud titled **Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics** https://arxiv.org/abs/1503.05049
4. Frieze paper by Emily Gunawan and Ralf Schiffler titled **Frieze vectors and unitary friezes**
   https://arxiv.org/abs/1806.00940

Thank you