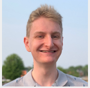
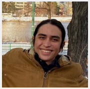
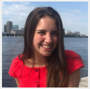
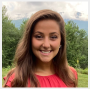
# The Box-Ball System (2020 UCONN REU)

BIRS Dynamical Algebraic Combinatorics Workshop

October 23, 2020

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# Tableaux

## Definition. (Young Tableaux)

- A Young tableau (pl. tableaux) is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- The Young tableau is *standard* if each row is an increasing sequence (going from left to right) and each column is an increasing sequence (going from top to bottom).
- The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

## Example. (Standard Young Tableau)

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array}$ is a standard Young tableau. Its reading word is 53412.
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \\ \cline{1-1} 5 & \\ \cline{1-1} 3 & \\ \cline{1-1} \end{array}$ is a nonstandard Young tableau.

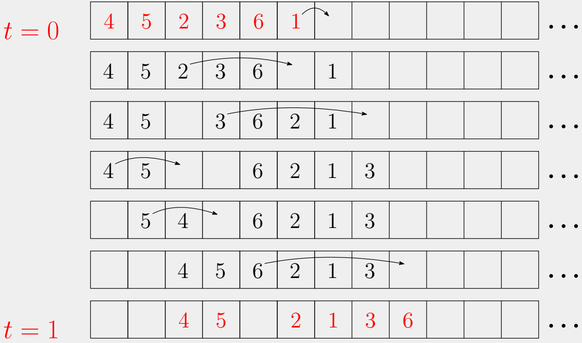
# Robinson-Schensted insertion algorithm

$$\pi = 452361 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = P$$

Let $P$ be the empty tableau. Given a permutation $\pi = \pi_1 \cdots \pi_n$, the *Robinson-Schensted (RS) insertion algorithm* is a method of insertion of each element of $\pi$ into $P$. The RS insertion algorithm always gives a standard Young tableau.

# Box-Ball System and Example

A box-ball system is a collection of discrete time states each containing elements of a permutation, $\pi$ in one-line notation. We can move forward in this system by rearranging $\pi$ using the following rule: Place $\pi$ in a strip of infinite boxes. (This corresponds to the $t = 0$ state of the system.) Then, move each element of $\pi$ to the nearest empty box to its right, as shown below.



# Box-Ball System: Example—$\pi = 452361$

To continue the analysis, keep making BBS moves until we reach a "steady state" (i.e., the system decomposes into *solitons*)

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 0$ | 4 | 5 | 2 | 3 | 6 | 1 | | | | | | | … | | | | | | |
| $t = 1$ | | | 4 | 5 | | 2 | 1 | 3 | 6 | | | | … | | | | | | |
| $t = 2$ | | | | | 4 | 5 | 2 | | | 1 | 3 | 6 | … | | | | | | |
| $t = 3$ | | | | | | | 4 | 2 | 5 | | | | 1 | 3 | 6 | | | … | |
| $t = 4$ | | | | | | | | 4 | | 2 | 5 | | | | | 1 | 3 | 6 | | … |

Then, create a (not necessarily standard) tableau by stacking solitons:

Soliton decomposition

| | | |
|---|---|---|
| 1 | 3 | 6 |
| 2 | 5 | |
| 4 | | |

with shape $(3, 2, 1)$.

**Theorem**

The following are equivalent:

1. $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$.
2. $\mathrm{SD}(\pi)$ is a standard tableau.

**Conjecture**

The following is equivalent to (1) and (2).

3. $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}\mathrm{P}(\pi)$.

- We have shown (1) $\iff$ (2) and (1) $\implies$ (3). It remains to prove (3) $\implies$ (1).

# Selected Results Involving Steady-State Times

**Theorem: Classification of permutations with steady-state value of $t = 0$**

A permutation $r$ has a box-ball steady-state value of $t = 0$ if and only if $r$ is the reading word of a standard tableau.

**Theorem: Some permutations with steady-state value of $t = 1$**

Let $r$ be the reading word of a standard tableau $P$. If $r'$ is the permutation resulting from the swapping of a special pattern of elements in $r$, then $r'$ has a box-ball steady-state value of $t = 1$. ($r'$ is the result of performing a *Knuth move* on $r$.)

**$n - 3$ conjecture**

The steady-state time for a permutation in $S_n$ is no more than $n - 3$.

Thank You!
Questions?