Type A and B c-Birkhoff polytopes

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Jt. w/

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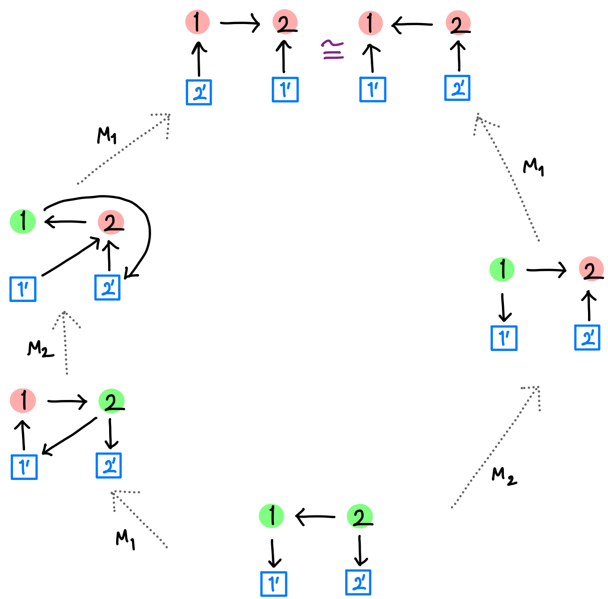
35th Meeting on the Representation Theory of Algebras & Related Topics

in honour of Thomas Brüstle's 60th birthday

Sat, 25 Oct 2025

ON MAXIMAL GREEN SEQUENCES

T. BRÜSTLE, G. DUPONT AND M. PÉROTIN



The maximal chains of $\overrightarrow{EG}(Q)$

$\longleftrightarrow$

maximal green sequences of Q

The oriented exchange graph

$\overrightarrow{EG}(Q)$ of $Q = 1 \leftarrow 2$

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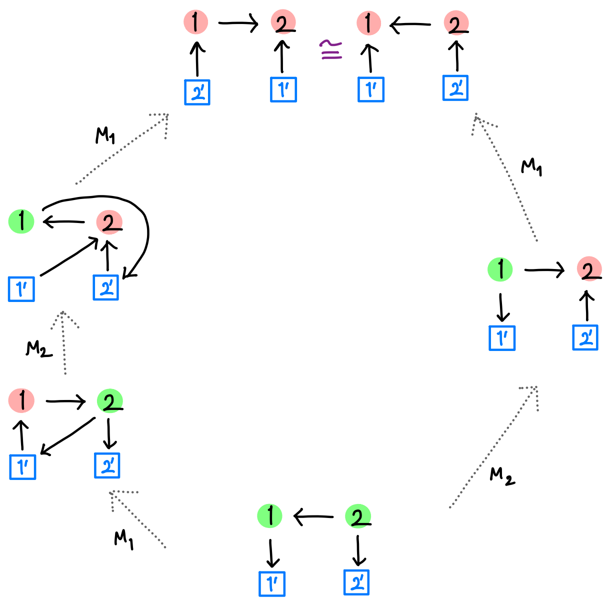


The oriented exchange graph $\overrightarrow{EG}(Q)$ of $Q = 1 \leftarrow 2$

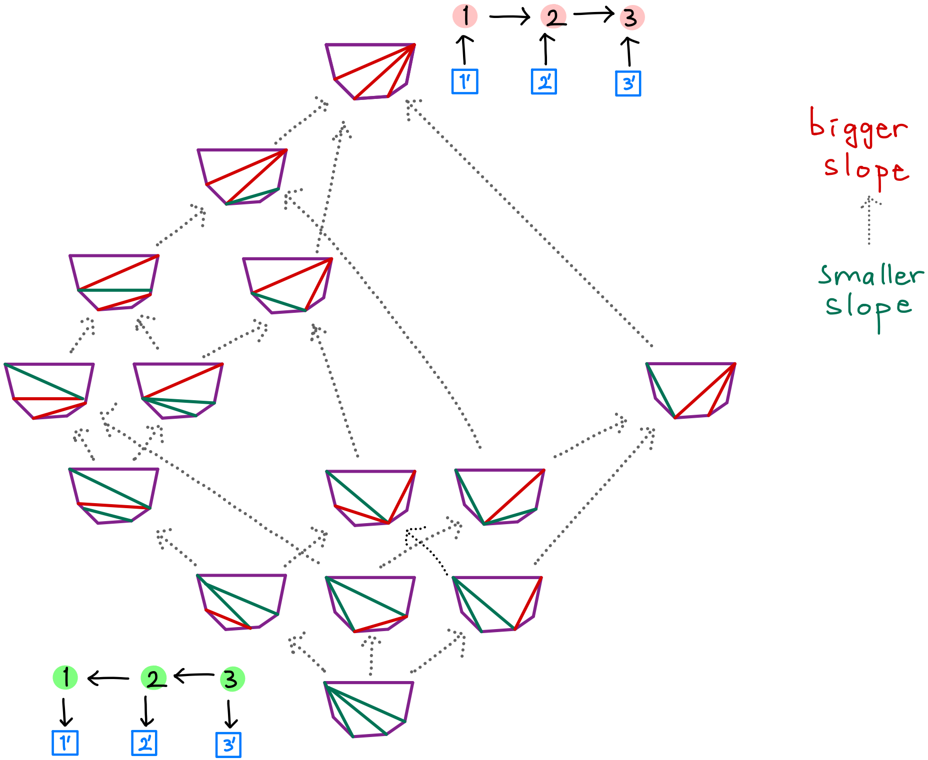
The maximal chains of $\overrightarrow{EG}(Q)$

$\longleftrightarrow$

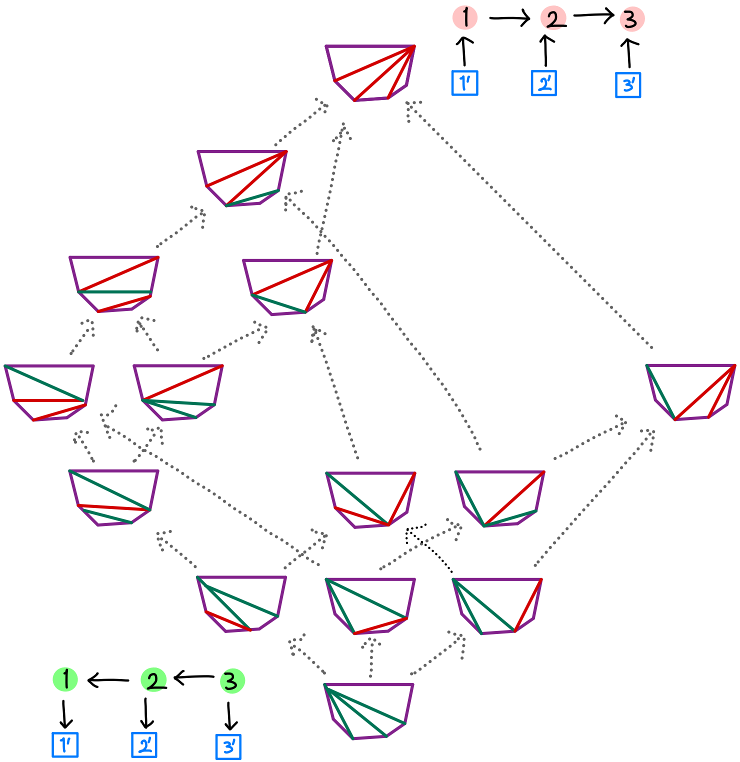
maximal green sequences of Q

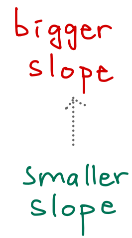
Question:

How many maximal green sequences of $Q = 1 \leftarrow 2$ are there?



The oriented exchange graph of $Q = 1 \leftarrow 2 \leftarrow 3$





The oriented exchange graph of $Q = 1 \leftarrow 2 \leftarrow 3$

Question:

How many maximal green sequences of $Q = 1 \leftarrow 2 \leftarrow 3$ are there?

Main Idea For each type A quiver Q, define a Birkhoff subpolytope $\mathrm{Birk}(Q)$ whose volume counts longest maximal green sequences of Q.

Note: # of longest maximal green sequences of Q

= # of linear extensions of a poset H_Q whose Hasse diagram is the Auslander-Reiten quiver of rep Q

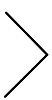
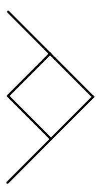
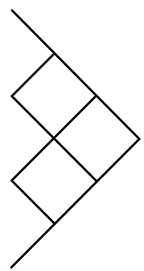
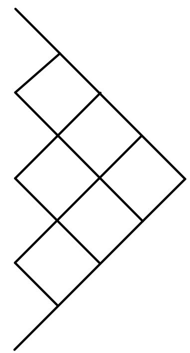
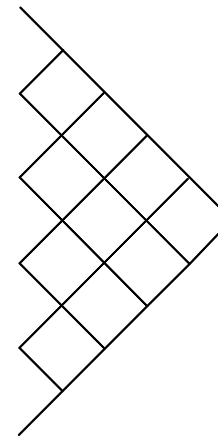
$$Q = \begin{tikzcd} & & 3 \arrow[dl] \arrow[dr] & \\ & 2 \arrow[dl] & & 4 \\ 1 & & & \end{tikzcd}$$

$$H_Q = \begin{tikzcd} & 3 & & \\ 34 \arrow[ur] & & 23 \arrow[ul] & \\ & 234 \arrow[ul] \arrow[ur] & & 123 \arrow[ul] \\ 2 \arrow[ur] & & P(3) = 1234 \arrow[ul] \arrow[ur] & \\ & P(2) = 12 \arrow[ul] \arrow[ur] & & P(4) = 4 \arrow[ul] \\ P(1) = 1 \arrow[ur] & & & \end{tikzcd}$$

Volume of $\mathrm{Birk}(Q) = 41$

# of linear extensions of $H_Q = 41$

Volume of $\mathrm{Birk}(Q)$ for $Q = 1 \leftarrow 2 \leftarrow 3 \cdots \leftarrow n$

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	
Q	$1 \leftarrow 2$	$1 \leftarrow 2 \leftarrow 3$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 4$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$	$1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6$	...
H_Q						...
Volume	1	2	12	286	33592	...

$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

OEIS A003121 $(1, 1, 1, 2, 12, 286, 33592, \ldots)$ counts:

(Fishel–Nelson '12)

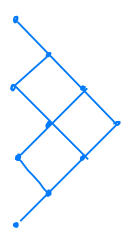
a. longest chains (length $\binom{n+1}{2}$) in the Tamari lattice,

i.e. the oriented exchange graph of $Q = 1 \leftarrow 2 \leftarrow 3 \cdots \leftarrow n$

b. Normalized volume of Conv(perm. matrices of 312, 132-avoiding permutations) (Davis–Sagan '16)

c. Linear extensions of the poset H whose Hasse diagram is the Auslander–Reiten quiver of rep$(\leftarrow \leftarrow \cdots \leftarrow)$

ex $n=4$



(OEIS 3rd comment '03)

d. Normalized volume of the order polytope $\mathcal{O}(H)$

I. Setup

$W = A_n$ the symmetric group S_{n+1}

generated by $\underbrace{s_1, \dots, s_n}_{\text{simple transpositions}}$ $\qquad s_k = (k \;\; k+1)$ cycle notation

c a Coxeter elt of W,

i.e. product of all n simple transpositions, in any order

$\updownarrow$ 1-1

Q an orientation of $\underset{1}{\bullet} - \underset{2}{\bullet} \cdots \bullet - \underset{n}{\bullet}$

Rule: $k \leftarrow \underline{k+1}$ if s_k is left of s_{k+1} in c ("in order") "lower-barred"

$k \rightarrow \overline{k+1}$ otherwise "upper-barred"

Ex:

A_4 gen'd by

$s_1 = (1\;2)$,

s_2

s_3

$s_4 = (4\;5)$

$c = \overbrace{s_1 s_4 s_2 s_3}^{\text{reduced word}}$

$= (1 \;\underline{2}\; \underline{3}\; 5\; \overline{4})$

$\updownarrow$

$Q = \text{Quiver}(c) =$

$$\begin{tikzcd}
 & & s_3 \arrow[dl, "\underline{3}"] \arrow[dr, "\overline{4}"] & \\
 & s_2 \arrow[dl, "\underline{2}"] & & s_4 \\
s_1 & & &
\end{tikzcd}$$

Heaps theory (Viennot '86, Stembridge '96, Stanley's EC Vol 1, ...)

Given a reduced word $u = s_{u_1} s_{u_2} \dots s_{u_\ell}$ of $w \in A_n$,

the <u>heap of u</u>, $\mathrm{Heap}(u)$, is the partial order $\preccurlyeq$ on $\{1, 2, \dots, \ell\}$ given by the transitive closure of the relations

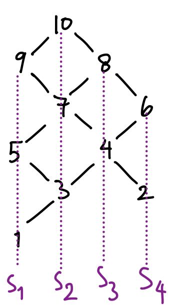
$$x \prec y$$

for $x < y$ such that $|u_x - u_y| \leq 1$.

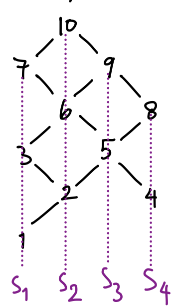
For each poset element $x \in \{1, 2, \dots, \ell\}$, the label of x is s_{u_x}

Ex: $u = s_1 s_4 s_2 s_3 \;\; s_1 s_4 s_2 s_3 \;\; s_1 s_2$, $\quad v = s_1 \;\; s_2 s_1 \;\; s_4 s_3 s_2 s_1 \;\; s_4 s_3 s_2$

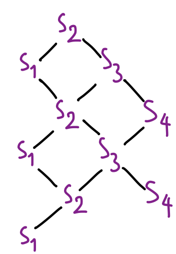
Heap(u):



Heap(v):



Heap(u) and Heap(v) have the same <u>heap diagram</u>:



* In general, if H is the heap of a reduced word $a_1 \cdots a_\ell$

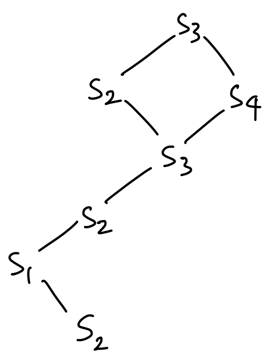
then $a_1 \cdots a_\ell$ is a <u>linear extension</u> of H, and

A total order π that is consistent w/ the structure of H, i.e.
$x \prec_H y$ implies $\pi(x) < \pi(y)$

$\{$linear extensions of $H\}$ = commutation class of $a_1 \cdots a_\ell$

(all words that can be obtained from $a_1 \cdots a_\ell$ by a sequence of commutation moves $s_i s_j \leftrightarrow s_j s_i$ for $|i-j| \geq 2$)

Ex: $\left\{ \text{Linear extensions of} \right.$



$\left. \right\}$ $=$ $\{ s_2\, s_1\, s_2\, s_3\, s_2\, s_4\, s_3,$

$s_2\, s_1\, s_2\, s_3\, s_4\, s_2\, s_3 \}$

* Def (N. Reading '07)

- $c^\infty := c \mid c \mid c \mid \cdots$

  Ex: If $c = s_1 s_4 s_2 s_3$ then $c^\infty = s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3 \mid \ldots$

- The <u>c-sorting word</u> of w is the lexicograhically first subword of c^∞ (as a sequence of positions in c^∞) that is a reduced word for w

  Notation: $\mathrm{sort}_c(w)$

  Ex: $s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3$ is a c-sorting word, $s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3$ is not

* Let w_0 denote the longest permutation, $(n+1) \cdots 3 2 1$
in 1-line notation

Then $\ell(w_0) = \binom{n+1}{2}$

Coxeter length

Ex: $w_0 = 54321$

$\ell(w_0) = \binom{5}{2} = 10$

Def Let H_Q be the following diagram:

Step 1 Draw the Auslander-Reiten quiver of rep Q vertically

Step 2 Replace indecomposables in the τ^{-1}-orbit of $P(j)$ with label s_j

$$Q = \begin{tikzcd}[column sep=small] & & 3 \arrow[dl] \arrow[dr] & \\ & 2 \arrow[dl] & & 4 \\ 1 & & & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & 3 & & \\ 34 \arrow[ur] & & 23 \arrow[ul] & \\ & 234 \arrow[ul] \arrow[ur] & & 123 \arrow[ul] \\ 2 \arrow[ur] & & P(3) = 1234 \arrow[ul] \arrow[ur] & \\ & P(2) = 12 \arrow[ul] \arrow[ur] & & P(4) = 4 \arrow[ul] \\ P(1) = 1 \arrow[ur] & & & \end{tikzcd}$$

$$H_Q = \begin{tikzcd}[column sep=small] & s_2 & & \\ s_1 \arrow[ur] & & s_3 \arrow[ul] & \\ & s_2 \arrow[ul] \arrow[ur] & & s_4 \arrow[ul] \\ s_1 \arrow[ur] & & s_3 \arrow[ul] \arrow[ur] & \\ & s_2 \arrow[ul] \arrow[ur] & & s_4 \arrow[ul] \\ s_1 \arrow[ur] & & & \end{tikzcd}$$

Then H_Q is the heap diagram for $\mathrm{sort}_c(w_0)$

$w_0 = 54321$ $\quad$ $\ell(w_0) = \binom{5}{2} = 10$

$\mathrm{sort}_c(w_0) = s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3 \mid s_1 s_2$ $\quad$ for $c = s_1 s_4 s_2 s_3$

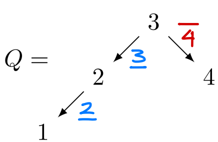
Alternatively, construct the heap diagram H_Q for $\mathrm{sort}_c(w_0)$ using lower- and upper-barred numbers:

(1) Draw a slope -1 "diagonal" D_{long}

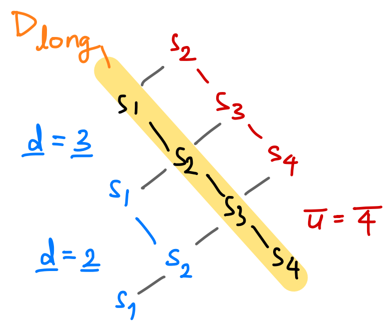
(2) Below D_{long}, for each $\underline{d}$, put a flushed-left diagonal $s_1 - s_2 \cdots - s_{d-1}$ w/ $d-1$ vertices

(3) Above D_{long}, for each $\overline{u}$, put a flushed-right diagonal $s_{n-u+2} \cdots s_{n-1} - s_n$ w/ $u-1$ vertices

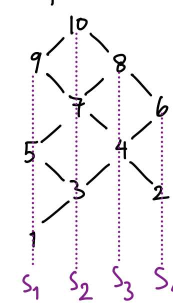
Ex:



$H_Q =$



Heap$(\mathrm{sort}_c(w_0)) =$



$\mathrm{sort}_c(w_0) = s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3 \mid s_1 s_2$ for $c = s_1 s_4 s_2 s_3$

II. The order polytope

The <u>order polytope</u> $\mathcal{O}(H)$ of a finite poset H is

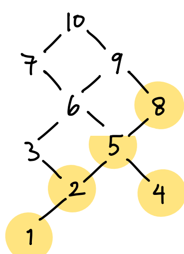
$$\mathcal{O}(H) = \left\{ \vec{x} \in \mathbb{R}^{|H|} : 0 \le \vec{x}(i) \le 1 \text{ for all } i = 1, \dots, |H| \right.$$
$$\left. \text{and } \vec{x}(i) \le \vec{x}(j) \text{ whenever } i \underset{\substack{|\\ \text{poset relation}}}{\lesssim} j \right\}$$

- $\{\text{Vertices of } \mathcal{O}(H)\} = \{\text{Indicator vectors of } I : I \text{ is an } \underset{\text{(down-set)}}{\underline{\text{order ideal}}} \text{ of } H\}$

(i.e. $\mathcal{O}(H)$ is the convex hull of the indicator vectors of order ideals of H)

Ex 1:

$I =$



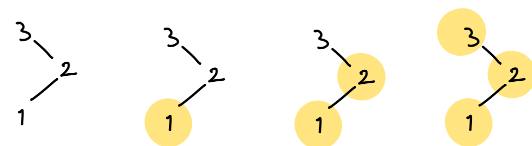
Indicator vector of $I = \begin{array}{c} 1 \\ 2 \\ \\ 4 \\ 5 \\ \\ \\ 8 \\ \\ \\ \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{10}$

II. The order polytope

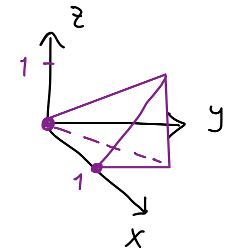
- $\{\text{Vertices of } \mathcal{O}(H)\} = \{\text{Indicator vectors of } I : I \text{ is an } \underline{\text{order ideal}} \text{ of } H\}$
  (down-set)

(i.e. $\mathcal{O}(H)$ is the convex hull of the indicator vectors of order ideals of H)

Ex 2:



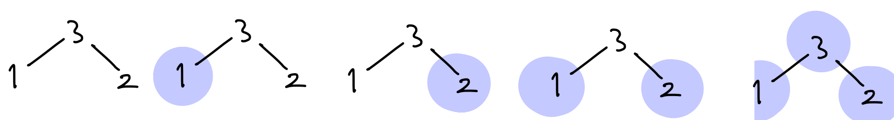
$$\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\end{bmatrix}$$



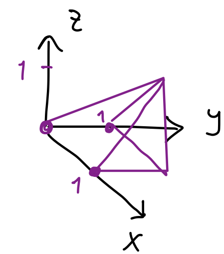
$$\text{Volume} = \frac{\text{base} \cdot \text{height}}{3} = \frac{1}{6}$$

$$(\text{dim})!\,(\text{volume}) = 3! \cdot \frac{1}{6} = 1$$

Ex 3:



$$\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\end{bmatrix}$$



$$\text{Volume} = \frac{\text{base} \cdot \text{height}}{3} = \frac{1}{3}$$

$$(\text{dim})!\,(\text{volume}) = 3! \cdot \frac{1}{3} = 2$$

II. The order polytope

Facts

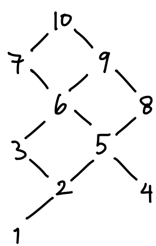
- $\dim \mathcal{O}(H) = |H|$

$\dim(\text{polytope})!\ \mathrm{Vol}(\text{polytope})$

- Normalized volume $= |\{\text{linear extensions of } H\}|$

Ex 1:

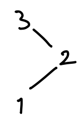
H =



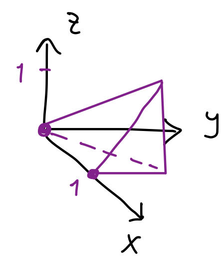
$\mathrm{Dim}\ \mathcal{O}(H) = 10$

N. Volume = 41

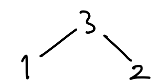
Ex 2:



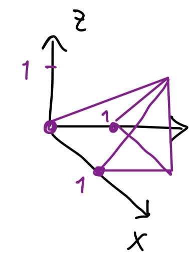
N. volume = 1



Ex 3:



N. volume = 2



$\boxed{\text{III. c-singletons}}$

TFAE:

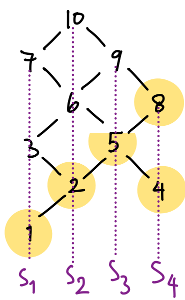
Def/Thm of (Hohlweg–Lange–Thomas '07) + heap theory

1. $w \in A_n$ is <u>c-singleton</u>

2. w corresponds to an order ideal I of $H = \mathrm{Heap}(\mathrm{sort}_c(w_0))$

i.e. w has a reduced word which is a linear extension of I.

Ex: $H=$ $I=$



$I = \{1, 2, 4, 5, 8\}$,

$w(I) = s_1 s_2 s_4 s_3 s_4$

3. w avoids four patterns $31\underline{2}$, $\overline{2}31$, $13\underline{2}$, $\overline{2}13$ (for all n)

Note: If $c = s_1 s_2 \cdots s_n$ then w is c-singleton iff w avoids 312, 132

$Q = 1 \leftarrow 2 \leftarrow 3 \cdots \leftarrow n$

IV. Our Birkhoff subpolytope

<u>Main Def</u>

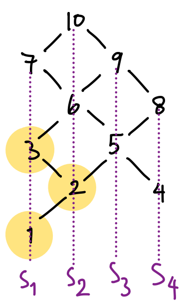
- $\mathrm{Birk}(c) := \mathrm{Conv}\left(\begin{array}{c}\text{permutation matrices } M(w) \\ \text{of } c\text{-singletons } w\end{array}\right)$

- The permutation matrix $M(w)$ of w is the $(n+1)\times(n+1)$ matrix s.t. row i, col j has entry $\begin{cases} 1 & \text{if } w(i)=j \\ 0 & \text{otherwise}\end{cases}$

Ex:

$H =$

$I =$



$I = \{1, 2, 3\}$,

$w(I) = s_1 s_2 s_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{bmatrix}$

2-line notation

$M(32145) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to$ vector in $\mathbb{R}^{25}$

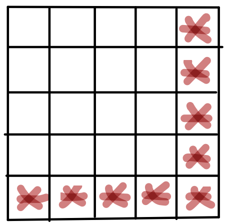
IV. Our Birkhoff subpolytope

Rem Birk(c) lives in $\mathbb{R}^{(n+1)^2} = \mathbb{R}^{25}$ but it has $\dim = \binom{n+1}{2} = \frac{(n+1)\,n}{2} = \frac{(5)4}{2} = 10$

Fifteen Relations that give us dim 10:

(Birkhoff relation)

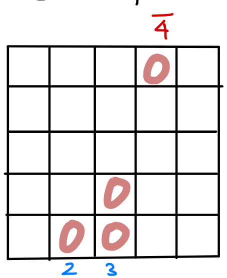
- Each row and col sum up to 1:



minus 9

(Zero relation)

- These four entries must be 0:

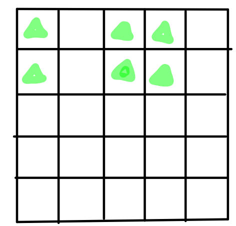


↳ from pattern-avoidance conditions

minus 4

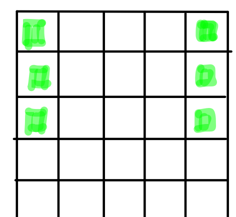
- (Summing relation) ↳ also from pattern-avoidance conditions

(i) These entries must sum up to 1:



minus 1

(ii) These entries must sum up to 1:



minus 1

+
___

minus 15

V. $\mathrm{Birk}(c) \cong \mathcal{O}(H)$

integrally equivalent

Define a lattice-preserving projection

$$\pi_c : \{\underbrace{5\times 5}_{(n+1)\times(n+1)} \text{ matrices in } \mathrm{Birk}(c)\} \longrightarrow \mathbb{R}^{10} \quad \mathbb{R}^{\frac{(n+1)n}{2}}$$

Rule:

$$\begin{array}{cccccc}
 & & & \bar{4} & \\
\left[\begin{array}{ccccc}
 & 10 & 8 & & 4 \\
 & & 9 & 1 & 5 \\
 & & & 2 & 6 \\
 & & & & 7 \\
3 & & & &
\end{array}\right] \\
 & \underline{2} & \underline{3} & &
\end{array}$$

$$\pi_c : M(32145) = \begin{array}{c}
\begin{array}{ccccc} & & & \bar{4} & \end{array} \\
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \\
\begin{array}{ccccc} & \underline{2} & \underline{3} & & \end{array}
\end{array} \longmapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thm

There exists an $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix U_c w/ 1's on the main diagonal (note: U_c has det 1) s.t. $U_c \circ \pi_c(M(w))$ = indicator vector of the order ideal I of H corresp. to w for all c-singletons w.

VI. $\mathrm{Birk}(c) \cong \mathcal{O}(H)$ in type B

Unfolding map $\lambda : B_n \to A_{2n-1}$

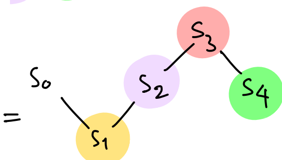
$\lambda :\ s_0 \mapsto s_n$

$\quad\ \ s_i \mapsto s_{n-i}\, s_{n+i}$ for $i > 0$

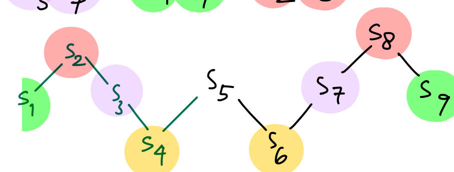
$c = s_1 s_0 s_2 s_4 s_3$ in B_5

$c' = s_4 s_6\ s_5\ s_3 s_7\ s_1 s_9\ s_2 s_8$ in A_9

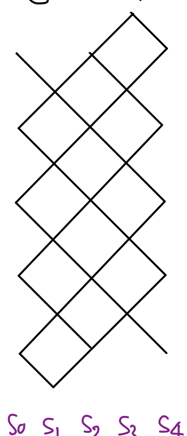
Heap(c) =



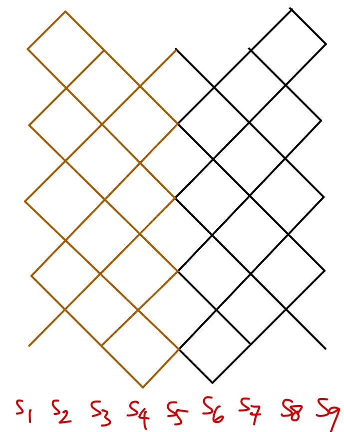
Heap(c') =



Heap diagram for $\mathrm{sort}_c(w_0)$ in B_5:



Heap diagram for $\mathrm{sort}_{c'}(w_0')$ in A_9



Further questions

Generalize to type BDEFGHI?

Further questions

ON MAXIMAL GREEN SEQUENCES

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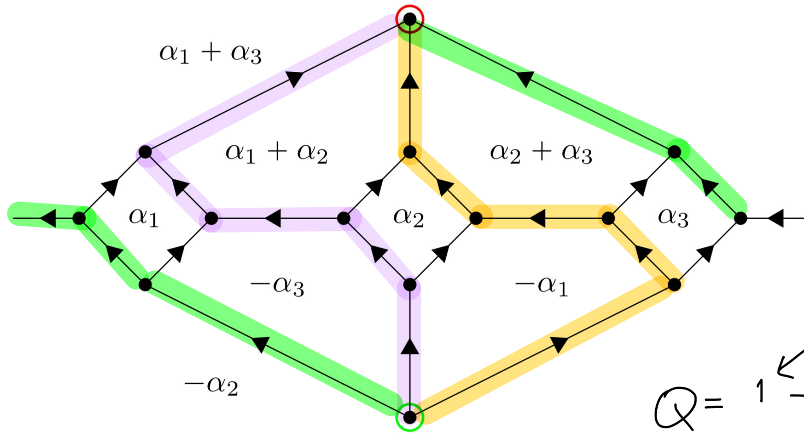


$$Q = \begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

FIGURE 4. The oriented exchange graph of the cyclic quiver with 3 vertices.

Let Q be mutation-equivalent to a type A Dynkin quiver.

Can we define a polytope whose normalized volume is # of longest maximal green sequences of Q?

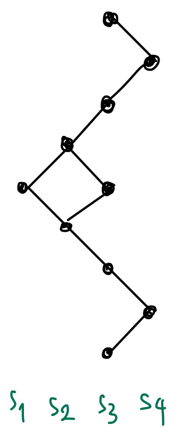
Further questions

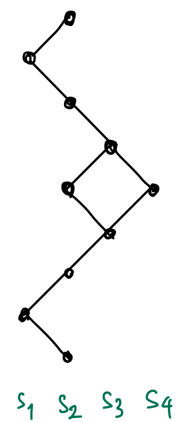
If u is a reduced word of and $H = \mathrm{Heap}(u)$, when is

$$\mathcal{O}(H) \cong \mathrm{Conv}(\text{perm. matrices of order ideals of } H)\ ?$$

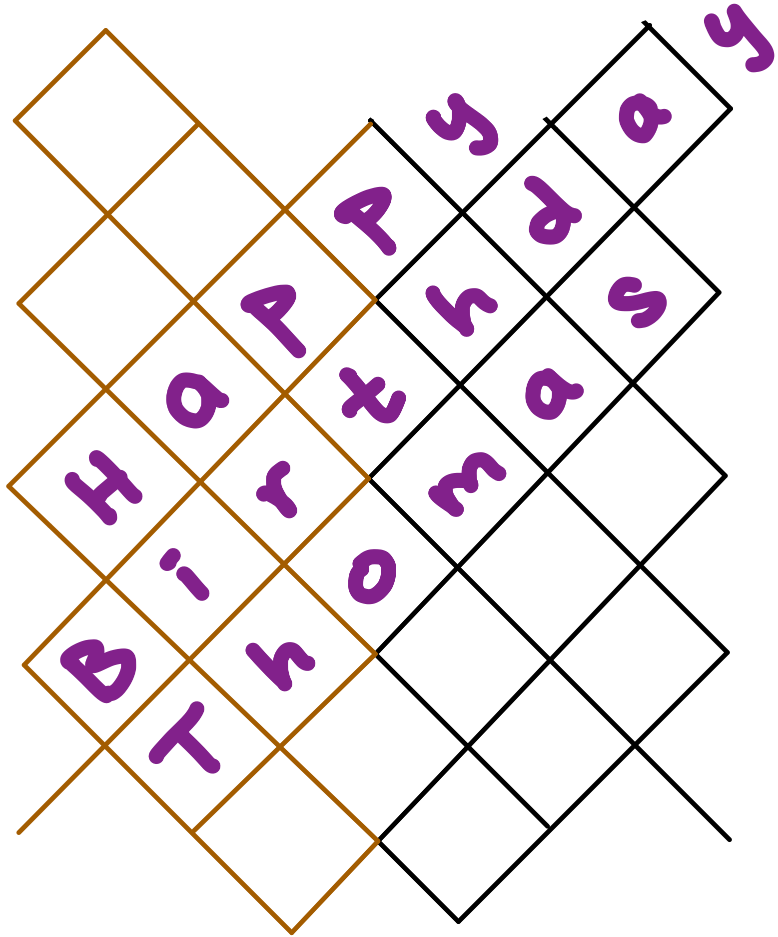
Example when the two polytopes aren't equivalent:

$s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4 s_3$, $\qquad$ $s_2 s_1 s_2 s_3 s_2 s_4 s_3 s_2 s_1 s_2$

$H =$ 

$H =$ 

$\dim \mathcal{O}(H) = 10$ and $\dim \mathrm{Birk}(H) = 9$, so $\mathcal{O}(H) \not\cong \mathrm{Birk}(H)$


Happy
Birthday
Thomas