Pattern-avoiding c-Birkhoff polytopes

Emily Gunawan (UMass Lowell)

Jt. w/

Esther Banaian, Sunita Chepuri, & Jianping Pan

CanaDAM 2025

CM 22: Recent advances in enumerative and geometric combinatorics on graphs

Wed, 21 May 2025

Main Result We define a "pattern-avoiding" polytope Birk(c) and prove that it is integrally equivalent to the (Stanley's) order polytope $\mathcal{O}(H)$, where H is the heap poset of the c-sorting word of the longest permutation w_0.

a Coxeter elt

Corollary

Volume of Birk(c) = # of linear extensions of H

= # of longest chains in the $\underbrace{\text{c-Cambrian lattice}}$

- its Hasse diagram is an oriented exchange graph of the cluster algebra with initial quiver Q = Quiver(c)

- a Tamari lattice if $c = s_1 s_2 \cdots s_n$
(a partial order on ways to use parentheses)

I. Setup

$W = A_n$ the symmetric group S_{n+1}

generated by $\underbrace{s_1, \dots, s_n}_{\text{simple transpositions}}$ $\quad s_k = (k \;\; k{+}1)$ cycle notation

c a Coxeter elt of W,

i.e. product of all n simple transpositions, in any order

$\updownarrow$ 1-1

Q an orientation of $\underset{1}{\bullet} - \underset{2}{\bullet} \cdots \bullet - \underset{n}{\bullet}$

Rule: $k \xleftarrow{\underline{k+1}} k{+}1$ if s_k is left of s_{k+1} in c ("in order") "lower-barred"

$k \xrightarrow{\overline{k+1}} k{+}1$ otherwise "upper-barred"

Ex: A_4 gen'd by

$s_1 = (1\;2)$,

s_2

s_3

$s_4 = (4\,5)$

$c = \overbrace{s_1 s_4 s_2 s_3}^{\text{reduced word}}$

$= (1 \; \underline{2} \; \underline{3} \; 5 \; \overline{4})$

$\updownarrow$

$Q = \text{Quiver}(c) =$

$$\begin{tikzcd} & & s_3 \arrow[dl, "\underline{3}"] \arrow[dr, "\overline{4}"] & \\ & s_2 \arrow[dl, "\underline{2}"] & & s_4 \\ s_1 & & & \end{tikzcd}$$

Q an orientation of $\underset{1}{\bullet} - \underset{2}{\bullet} \cdots \bullet - \underset{n}{\bullet}$

Rule: $k \leftarrow k+1$ if s_k is left of s_{k+1} in c ("in order")

$k \rightarrow k+1$ otherwise

Heap(c) is Quiver(c), thought of as a Hasse diagram labeled by the s_k

Ex:

$Q = \text{Quiver}(c) =$

$$\begin{tikzcd} & & s_3 \arrow[dl, "\underline{3}"'] \arrow[dr, "\overline{4}"] & \\ & s_2 \arrow[dl, "\underline{2}"'] & & s_4 \\ s_1 & & & \end{tikzcd}$$

$\Downarrow$ Remove orientation

$$\begin{tikzcd} & & s_3 \arrow[dl, no head] \arrow[dr, no head] & \\ & s_2 \arrow[dl, no head] & & s_4 \\ s_1 & & & \end{tikzcd}$$

Heap(c)

$s_1 s_4 s_2 s_3$
$s_1 s_2 s_4 s_3$
$s_4 s_1 s_2 s_3$

Note:
All reduced words of c are linear extensions of C

* In general, if H is the heap of a reduced word $a_1 \cdots a_\ell$

then $a_1 \cdots a_\ell$ is a <u>linear extension</u> of H, and

A total order π that is consistent w/ the structure of H, i.e.

$x \underset{H}{<} y$ implies $\pi(x) < \pi(y)$

$\{$linear extensions of $H\}$ = commutation class of $a_1 \cdots a_\ell$

(all words that can be obtained from $a_1 \cdots a_\ell$ by a sequence of commutation moves $s_i s_j \leftrightarrow s_j s_i$ for $|i-j| \geq 2$)

* Def (N. Reading '07)

- $c^\infty := c \,|\, c \,|\, c \,|\, \cdots$

  Ex: If $c = s_1 s_4 s_2 s_3$ then $c^\infty = s_1 s_4 s_2 s_3 \,|\, s_1 s_4 s_2 s_3 \,|\, s_1 s_4 s_2 s_3 \,|\, \ldots$

- A <u>c-sorting word</u> of w is the lexicograhically first subword of c^∞ (as a sequence of positions in c^∞) that is a reduced word for w

  Notation: $\mathrm{sort}_c(w)$

  Ex: $s_1 s_4 s_2 s_3 \,|\, s_1 s_4 s_2 s_3$ is a c-sorting word, $s_1 s_4 s_2 s_3 \,|\, s_1 s_4 s_2 s_3$ is not

w_0 longest permutation, $(n+1)\cdots 321$ in 1-line notation

$\ell(w_0) = \binom{n+1}{2}$

Coxeter length

$w_0 = 54321$

$\ell(w_0) = \binom{5}{2} = 10$

Ex: $\underbrace{s_1 s_4 s_2 s_3 \mid s_1 s_4 s_2 s_3 \mid s_1 s_2}_{\text{sort}_c(w_0)}$ if $c = s_1 s_4 s_2 s_3$

Prop The heap H of $\text{sort}_c(w_0)$ is as follows:

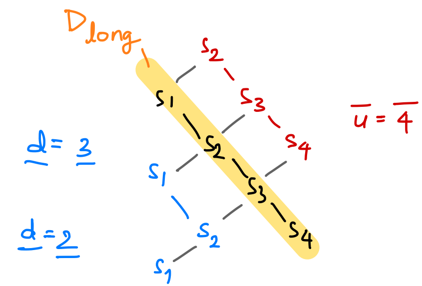
(1) Draw a slope -1 "diagonal" D_{long}

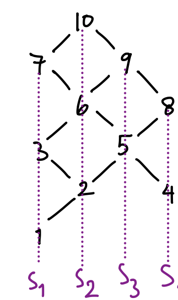
(2) Below D_{long}, for each d, put a flushed-left diagonal

$s_1 \diagdown s_2 \cdots \diagdown s_{d-1}$

(3) Above D_{long}, for each u, put a flushed-right diagonal w/ $u-1$ vertices

$s_{n-u+2} \cdots s_{n-1} \diagdown s_n$



$H =$ 

* The elts of H are $\{1, 2, \ldots, \ell(w_0)\}$, read following the diagonals bottom to top

II. The order polytope

The order polytope $\mathcal{O}(H)$ of a finite poset H is

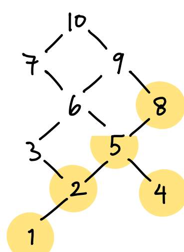
$$\mathcal{O}(H) = \left\{ \vec{x} \in \mathbb{R}^{|H|} : 0 \le \vec{x}(i) \le 1 \text{ for all } i = 1, \dots, |H| \text{ and } \vec{x}(i) \le \vec{x}(j) \text{ whenever } i \underset{\text{poset relation}}{\lesssim} j \right\}$$

Facts

- $\{\text{Vertices of } \mathcal{O}(H)\} = \{\text{Indicator vectors of } I : I \text{ is an order ideal (down-set) of } H\}$

(i.e. $\mathcal{O}(H)$ is the convex hull of the indicator vectors of order ideals of H)

Ex:

$I =$ 

$$\text{Indicator vector of } I = \begin{array}{c} 1 \\ 2 \\ \\ 4 \\ 5 \\ \\ \\ 8 \\ \\ \\ \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{10}$$

II. The order polytope

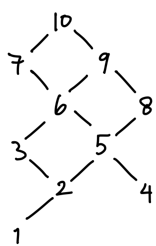
The **order polytope** $\mathcal{O}(H)$ of a finite poset H is

$$\mathcal{O}(H) = \left\{ \vec{x} \in \mathbb{R}^{|H|} : 0 \leq \vec{x}(i) \leq 1 \text{ for all } i = 1, \dots, |H| \right.$$
$$\left. \text{and } \vec{x}(i) \leq \vec{x}(j) \text{ whenever } i \underset{\text{poset relation}}{\preceq} j \right\}$$

Facts

- $\dim \mathcal{O}(H) = |H|$
- $\overbrace{\text{Normalized volume}}^{\dim(\text{polytope})!\ \mathrm{Vol}(\text{polytope})} = \left| \{ \text{linear extensions of } H \} \right|$

Ex:



$\mathrm{Dim}\ \mathcal{O}(H) = 10$

$\mathrm{Volume} = 41$

III. c-singletons

TFAE:

Def/Thm of (Hohlweg–Lange–Thomas '07) + heap theory

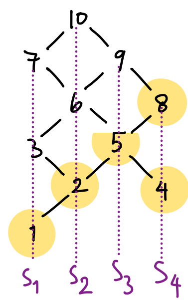
1. $w \in A_n$ is <u>c-singleton</u>

2. w corresponds to an order ideal I of H,

i.e. w has a reduced word which is a linear extension of I.

Ex: $H =$

$I =$



$I = \{1, 2, 4, 5, 8\}$, $\quad w(I) = s_1 s_2 s_4 s_3 s_4$

3. w avoids four patterns $31\underline{2}$, $\overline{2}31$, $13\underline{2}$, $\overline{2}13$ (for all n)

Rem: If $c = s_1 s_2 \cdots s_n$ then w is c-singleton iff w avoids 312, 132

$Q = 1 \leftarrow 2 \leftarrow \cdots$

IV. A "pattern-avoiding" Birkhoff subpolytope

Def (Our pattern-avoiding polytope):

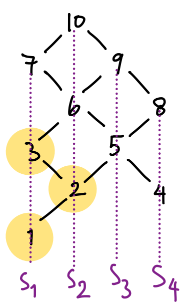
- $\mathrm{Birk}(c) := \mathrm{Conv}\left(\begin{array}{c}\text{permutation matrices } M(w) \\ \text{of } c\text{-singletons } w\end{array}\right)$

- The permutation matrix $M(w)$ of w is

the $(n+1)\times(n+1)$ matrix s.t. row i, col j has entry $\begin{cases} 1 & \text{if } w(i)=j \\ 0 & \text{otherwise}\end{cases}$

Ex:

$H =$



$I =$

$I = \{1, 2, 3\}$,

$w(I) = s_1 s_2 s_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{bmatrix}$

2-line notation

$M(32145) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to$ vector in $\mathbb{R}^{25}$

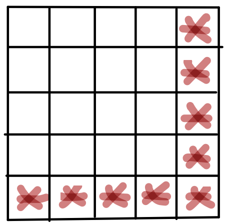
IV. A "pattern-avoiding" Birkhoff subpolytope

Rem Birk(C) lives in $\mathbb{R}^{(n+1)^2} = \mathbb{R}^{25}$ but it has $\dim = \binom{n+1}{2} = \frac{(n+1)n}{2} = \frac{(5)4}{2} = 10$

Fifteen Relations that give us dim 10:

(Birkhoff relation)

- Each row and col sum up to 1:

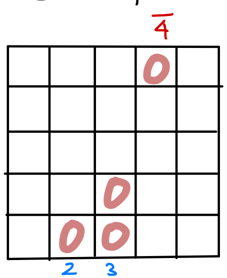


minus

9

(Zero relation)

- These four entries must be 0:



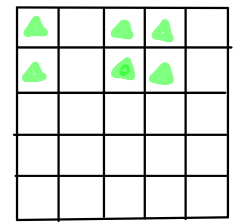
↳ from pattern-avoidance conditions

minus

4

- (Summing relation) ↳ also from pattern-avoidance conditions

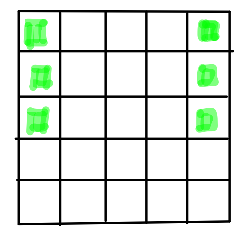
(i) These entries must sum up to 1:



minus

1

(ii) These entries must sum up to 1:



minus

1

+

minus 15

V. $\mathrm{Birk}(c) \cong \mathcal{O}(H)$

$\cong$: integrally equivalent

Define a lattice-preserving projection

$$\pi_c : \{\underbrace{5\times 5}_{(n+1)\times(n+1)} \text{ matrices in } \mathrm{Birk}(c)\} \longrightarrow \mathbb{R}^{10} \quad \left(\mathbb{R}^{\frac{(n+1)n}{2}}\right)$$

Rule:

$$\begin{array}{c|c|c|c|c|c|}
 & & & & \overline{4} & \\
\hline
 & & 10 & 8 & & 4 \\
\hline
 & & & 9 & 1 & 5 \\
\hline
 & & & & 2 & 6 \\
\hline
 & & & & & 7 \\
\hline
 & 3 & & & & \\
\hline
 & & \underline{2} & \underline{3} & &
\end{array}$$

$$\pi_c : M(32145) = \begin{array}{ccccc}
 & & & \overline{4} & \\
\left[\begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix}\right] \\
 & \underline{2} & \underline{3} & &
\end{array} \longmapsto \begin{array}{c} 1 \\ 2 \\ \\ 4 \\ 5 \\ \\ \\ 8 \\ \\ \\ \end{array} \left[\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{matrix}\right]$$

Thm

There exists an $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix $\mathcal{U}_c$ w/ 1's on the main diagonal (note: $\mathcal{U}_c$ has det 1) s.t. $\mathcal{U}_c \circ \pi_c(M(w))$ = indicator vector of the order ideal I of H corresp. to w for all c-singletons w.

VI. Inspiration

n= 0 1 2 3 4 5 6

OEIS A003121 (1, 1, 1, 2, 12, 286, 33592, ...)

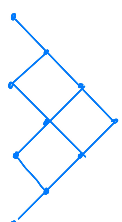
counts:

a. longest chains (length $\binom{n+1}{2}$) in the Tamari lattice (Fishel – Nelson '12)

b. Normalized volume of Conv(perm. matrices of 312, 132 - avoiding permutations) (Davis – Sagan '16)

c. Linear extensions of the poset H whose Hasse diagram is the Auslander–Reiten quiver of rep$(\leftarrow \leftarrow \cdots \leftarrow)$

ex $n=4$



(OEIS 3rd comment '03)

d. Normalized volume of the order polytope $\mathcal{O}(H)$

VII. Questions

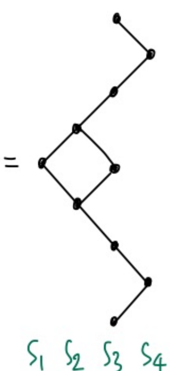
1. Our H is the heap of the c-sorting word of w_0.
If we take an arbitrary reduced word $[u]$ of an arbitrary permutation w, and let $H = \mathrm{Heap}([u])$, when is $\mathcal{O}(H) \cong \mathrm{Conv}(\text{perm. matrices of order ideals of } H)$?

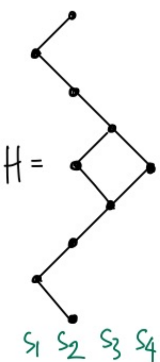
Ans Not in general.

Two counterexamples in A_4:

More counterex in A_5 and bigger A_n

$[u] = [3\ 4\ 3\ 2\ 3\ 1\ 2\ 3\ 4\ 3]$ $[u] = [2\ 1\ 2\ 3\ 2\ 4\ 3\ 2\ 1\ 2]$

$H =$ 

$H =$ 

$\dim \mathcal{O}(H) = 10$, $\dim \mathrm{Birk}(H) = 9$, so $\mathcal{O}(H) \not\cong \mathrm{Birk}(H)$.

VII. Questions

2. Relate our results to rep theory meaningfully

3. Generalize to type BDEFGHI ?

linear extensions

A_4 41 our ex

12 $\cdot \to \cdot \to \to$ OEIS A003121

70 $\to \leftarrow \to$