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Pattern-avoiding c-Birkhoff polytopes

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Main Result We define a "pattern-avoiding" polytope Birk(c) (a Coxeter elt) and prove that it is integrally equivalent to the (Stanley's) order polytope $\mathcal{O}(H)$, where H is the heap poset of the c-sorting word of the longest permutation w_0.

Corollary

Volume of Birk(c) = # of linear extensions of H

= # of longest chains in the c-Cambrian lattice

- its Hasse diagram is an oriented exchange graph of the cluster algebra with initial quiver Q = Quiver(c)

- a Tamari lattice if $c = s_1 s_2 \dots s_n$
(a partial order on ways to use parentheses)

I. Setup

$W = A_n$ the symmetric group S_{n+1}

generated by $\underbrace{s_1, \dots, s_n}_{\text{simple transpositions}}$

$s_k = (k \;\; k+1)$
cycle notation

c a Coxeter elt of W,

i.e. product of all n simple transpositions, in any order

$\updownarrow$ 1-1

Choosing a Coxeter elt is equivalent to choosing an orientation of the type A_n Dynkin diagram, the path graph w/ n vertices

Q an orientation of $\underset{1}{\bullet} — \underset{2}{\bullet} \cdots \bullet — \underset{n}{\bullet}$

Rule: $k \leftarrow k+1$ if s_k is left of s_{k+1} in c ("in order")

$k \rightarrow k+1$ otherwise

Heap(c) is Quiver(c), thought of as a Hasse diagram labeled by the s_k.

upper-barred numbers $\underline{u}$
in decreasing order

$c = (1 \; \boxed{} \; (n+1) \; \boxed{} \;)$

lower-barred numbers $\underline{d}$
in increasing order

Ex:

A_4 gen'd by
$s_1 = (1\ 2)$,
s_2
s_3
$s_4 = (4\ 5)$

$c = \overbrace{s_1 s_4 s_2 s_3}^{\text{reduced word}}$

$= (1\ \underline{2}\ \underline{3}\ 5\ \overline{4})$

$\updownarrow$

$Q = $ Quiver$(c) =$

$$\begin{tikzcd} & & s_3 \arrow[dl, "\underline{3}"] \arrow[dr, "\overline{4}"] & \\ & s_2 \arrow[dl, "\underline{2}"] & & s_4 \\ s_1 & & & \end{tikzcd}$$

$\Downarrow$ Remove orientation

$$\begin{tikzcd} & & s_3 \arrow[dl, dash] \arrow[dr, dash] & \\ & s_2 \arrow[dl, dash] & & s_4 \\ s_1 & & & \end{tikzcd}$$

Heap(c)

$s_1 s_4 s_2 s_3$
$s_1 s_2 s_4 s_3$
$s_4 s_1 s_2 s_3$

Note:
All reduced words of c are linear extensions of C

In general, if H is the heap of a reduced word $a_1 \cdots a_\ell$ then $a_1 \cdots a_\ell$ is a <u>linear extension</u> of H, and

A total order π that is consistent w/ the structure of H, i.e. $x \lessdot_H y$ implies $\pi(x) < \pi(y)$

$\{$linear extensions of $H\} = \{$commutation class of $a_1 \cdots a_\ell\}$

(all words that can be obtained from $a_1 \cdots a_\ell$ by a sequence of commutation moves $s_i s_j \leftrightarrow s_j s_i$ for $|i-j| \geq 2$.

Def (N. Reading '07)

A <u>c-sorting word</u> of w is the lexicograhically first (as a sequence of positions in $c^\infty := c \,|\, c \,|\, c \,|\, \cdots$) subword of c^∞ that is a reduced word for w.

Notation: $\text{sort}_c(w)$

w_0 longest permutation, $(n+1)\cdots 321$ in 1-line notation, $\ell(w_0) = \binom{n+1}{2}$

Coxeter length

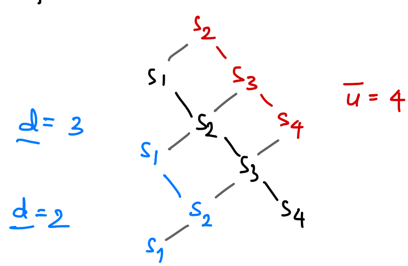
$w_0 = 54321$

$\ell(w_0) = \binom{5}{2} = 10$

(I'm more interested in the heap of $\mathrm{sort}_c(w_0)$ than in the word $\mathrm{sort}_c(w_0)$ itself)

<u>Prop</u> The heap H of $\mathrm{sort}_c(w_0)$ is as follows:

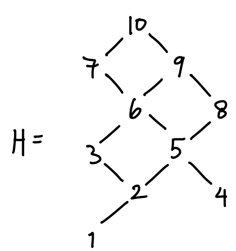
(1) Draw a slope -1 "diagonal" D_{long}



(2) Below D_{long}, for each $\underline{d}$,

put a flushed-left diagonal

$$s_1 \diagdown s_2 \;\cdots\; \diagdown s_{d-1}$$



(3) Above D_{long}, for each $\underline{u}$,

put a flushed-right diagonal

w/ $u-1$ vertices

$$s_{n-u+2} \;\cdots\; s_{n-1} \diagdown s_n$$

* The elts of H are $\{1, 2, \ldots, \ell\}$, read following the diagonals bottom to top

Note:

Rotating H by $90°$ clockwise gives us the "combinatorial AR quiver" for $\mathrm{sort}_c(w_0)$. This terminology is because it is isomorphic to the Auslander–Reiten quiver of Quiver(c).

Ex:

$$Q = \text{Quiver}(c) = \begin{tikzcd} & & 3 \arrow[dl] \arrow[dr] & \\ & 2 \arrow[dl] & & 4 \\ 1 & & & \end{tikzcd}$$

$$\Gamma = \text{AR quiver of rep } Q = \begin{tikzcd}[column sep=small] P_1 = 1 \arrow[dr] & & 2 \arrow[dr] & & \begin{smallmatrix}3\\4\end{smallmatrix} = I_4 \arrow[dr] & \\ & P_2 = \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 = I_3 \\ & & P_3 = \begin{smallmatrix}&3&\\2&&4\\1&&\end{smallmatrix} \arrow[ur] \arrow[dl] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} = I_2 \arrow[ur] & \\ & P_4 = 4 & & \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix} = I_1 \arrow[ur] & & \end{tikzcd}$$

$$H = \begin{tikzcd}[column sep=small, row sep=small] & & 10 \arrow[dl, dash] \arrow[dr, dash] & & \\ & 7 \arrow[dr, dash] & & 9 \arrow[dl, dash] \arrow[dr, dash] & \\ & & 6 \arrow[dl, dash] \arrow[dr, dash] & & 8 \arrow[dl, dash] \\ & 3 \arrow[dr, dash] & & 5 \arrow[dl, dash] \arrow[dr, dash] & \\ & & 2 \arrow[dl, dash] & & 4 \\ & 1 & & & \end{tikzcd}$$

II. The order polytope

The <u>order polytope</u> $\mathcal{O}(H)$ of a finite poset H is

$$\mathcal{O}(H) = \{ \vec{x} \in \mathbb{R}^{|H|} : 0 \leq \vec{x}(i) \leq 1 \text{ for all } i = 1, \dots, |H| \text{ and } \vec{x}(i) \leq \vec{x}(j) \text{ whenever } \underbrace{i \lesssim j}_{\text{poset relation}} \}$$

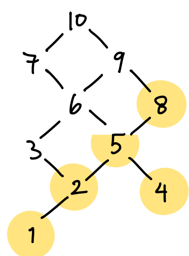
<u>Facts</u>

- $\dim \mathcal{O}(H) = |H|$
- $\{\text{Vertices of } \mathcal{O}(H)\} = \{\text{Indicator vectors of } I : I \text{ is an order ideal of } H\}$ (down-set)

(i.e. $\mathcal{O}(H)$ is the convex hull of the indicator vectors of order ideals of H)

Ex:

$I =$



$$\text{Indicator vector of } I = \begin{array}{c} 1 \\ 2 \\ \\ 4 \\ 5 \\ \\ \\ 8 \\ \\ \\ \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- $\underbrace{\text{Normalized volume}}_{\dim(\text{polytope})!\ \text{Vol}(\text{polytope})} = |\{\text{linear extensions of } H\}|$

Ex: $\dim \mathcal{O}(H) = 10$,

Volume $= 41$

III. c-singletons

TFAE:

1. w is <u>c-singleton</u>

Def / Thm of (Hohlweg – Lange – Thomas '07)

pronounced like "Hol weg" "Lang-e"

\+ heap theory

2. w corresponds to an order ideal I of H,

i.e w has a reduced word which is a linear extension of I.

Ex: $I = \{1, 2, 4, 5, 8\}$

$$w(I) = s_1 s_2 s_4 s_3 s_4$$

3. w avoids four certain patterns (for all n)

$31\underline{2}$, $\bar{2}31$, $13\underline{2}$, $\bar{2}13$ (Reading '04)

Rem:

If $c = s_1 s_2 \ldots s_n$ then w is c-singleton iff w avoids 312, 132

$$Q = 1 \leftarrow 2 \leftarrow \cdots$$

(the "Tamari case")

"the four patterns collapse to just two patterns"

IV. A "pattern-avoiding" Birkhoff subpolytope

Def (Our pattern-avoiding polytope):

- $\mathrm{Birk}(c) := \mathrm{Conv}\begin{pmatrix} \text{permutation matrices } M(w) \\ \text{of } c\text{-singletons } w \end{pmatrix}$

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \;\middle|\; \sum_{i=1}^{m} \lambda_i = 1 \;,\; \lambda_i \geq 0,\ x_i \in X \right\}$$

- The permutation matrix $M(w)$ of w is

the $(n \times 1) \times (n+1)$ matrix s.t. row i, col j has entry $\begin{cases} 1 & \text{if } w(i) = j \\ 0 & \text{otherwise} \end{cases}$

Ex: $I = \{1, 2, 3\}$, $\quad w(I) = s_1 s_2 s_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$ 2-line notation

$$M(32145) = \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline & 2 & & & \\ \hline 1 & & & & \\ \hline & & & 4 & \\ \hline & & & & 5 \\ \hline \end{array} \longrightarrow \text{vector in } \mathbb{R}^{25}$$

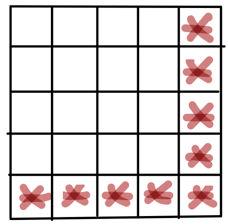
__Rem__ Birk(C) lives in $\mathbb{R}^{(n+1)^2} = \mathbb{R}^{25}$

__Prop__ Birk(C) has $\dim = \binom{n+1}{2} = \frac{(n+1)\,n}{2} = \frac{(5)4}{2} = 10$

Fifteen Relations that give us dim 10:

(Birkhoff relation)

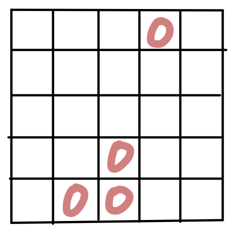
- Each row and col sum up to 1:



minus 9

(Zero relation)

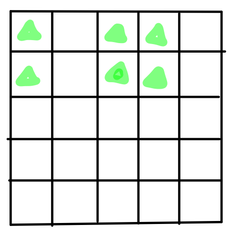
- The pattern avoidance conditions immediately require these four entries to be 0:



minus 4

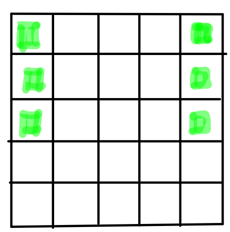
- (Summing relation)

(i) These entries must sum up to 1:



minus 1

(ii) These entries must sum up to 1:



minus 1

+ minus 15

V. $\mathrm{Birk}(c) \cong \mathcal{O}(H)$

integrally equivalent

Projection Π_c : $(n+1)\times(n+1)$ 5×5 matrices in $\mathrm{Birk}(c) \longrightarrow \mathbb{R}^{\binom{n+1}{2}}$ $\mathbb{R}^{10}$

$c = (\bar{4}\ 1\ \underline{2}\ \underline{3}\ 5)$ $\qquad Q = \mathrm{Quiver}(c) = 1 \swarrow 2 \swarrow 3 \searrow 4$ (vertices $\underline{2}$, $\underline{3}$, $\bar{4}$)

$$\begin{array}{c|ccccc|}
 & \underline{2} & \underline{3} & & & \\
\hline
 & \cdot & 10 & 8 & 0 & 4 \\
 & & \cdot & 9 & 1 & 5 \\
 & & & \cdot & 2 & 6 \\
 & & & 0 & \cdot & 7 \\
 & 3 & 0 & 0 & & \cdot \\
\hline
 & \text{also} & & & \bar{4} &
\end{array}$$

Do: col $\underline{2}$: $(1,\underline{2})^{(10)}$

col $\underline{3}$: $(2,\underline{3})^{(9)}, (1,\underline{3})^{(8)}$

col $n+1$: $(4,5)^{(7)}, (3,5)^{(6)}, (2,5)^{(5)}, (1,5)^{(4)}$

Then do: $u = \bar{4}$, $m = \min(u-1, n+1-u) = \min(3, 5-4) = 1$

- First: $\underbrace{(n+1, c^1(u))}_{(5,\ c(4)) = (5,1)^{(3)}}, (n, c^2(u)), \dots, (n+2-m, c^m(u))$

- Then, if $u-1 > m$, do: $\overset{4}{u} - 1 - \overset{1}{m} = 2$ entries:

$\underbrace{(u-1, u), (u-2, u)}_{}, \dots, (m+1, u)$

$(3,4)^{(2)}, (2,4)^{(1)}$

Thm Π_c sends integral points to integral points

"$\mathrm{Birk}(c)$ has no internal integral points"

Thm Fix Coxeter elt c.

There exists a unique $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix $\mathcal{U}_c$ with 1's on the main diagonal (in particular, $\mathcal{U}_c$ has det 1)

s.t $\mathcal{U}_c \circ \Pi_c(M(w))$

$=$ indicator vector of the order ideal I of H corresponding to w

for all c-singletons w. $\square$

VI. Inspiration

$n = \; 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

OEIS A003121 $(1, 1, 1, 2, 12, 286, 33592, \ldots)$ counts:

a. longest chains (length $\binom{n+1}{2}$) in Tamari lattice

(Fishel-Nelson '12)

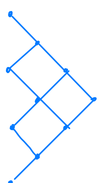
b. (normalized) volume of $B = \mathrm{Conv}($perm. matrices avoiding 312 and 132$)$

(Davis - Sagan '16)

c. linear extensions of poset H whose Hasse diagram is the Auslander-Reiten quiver of $\mathrm{rep}(\leftarrow \leftarrow \cdots \leftarrow)$

(OEIS 3rd comment '03)

ex $n=4$



d. (normalized) volume of the order polytope $\mathcal{O}(H)$

VII. Questions

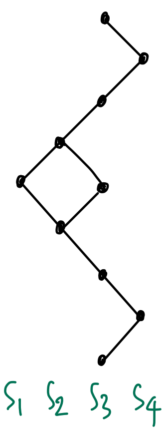
(1.) Our H is the heap of the c-sorting word of w_0.
If we take an arbitrary reduced word $[u]$ of an arbitrary permutation w, and let $H = \mathrm{Heap}([u])$, when is $\mathcal{O}(H) \cong \mathrm{Conv}($perm. matrices of order ideals of $H)$?

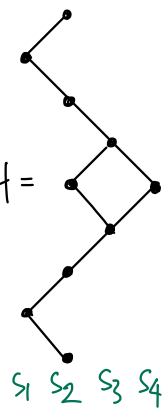
Ans Not in general.

Two counterexamples in A_4:

More counterex in A_5 and bigger A_n

$[u] = [3\ 4\ 3\ 2\ 3\ 1\ 2\ 3\ 4\ 3]$ $\quad$ $[u] = [2\ 1\ 2\ 3\ 2\ 4\ 3\ 2\ 1\ 2]$

$H =$ 

$H =$ 

$\dim \mathcal{O}(H) = 10$, $\dim \mathrm{Birk}(H) = 9$, so $\mathcal{O}(H) \ncong \mathrm{Birk}(H)$.

(2.) Relate our results to rep theory meaningfully

(3.) Generalize to type BDEFGHI ?

linear extensions

A_4 41 our ex

12 $\cdot\!\to\!\cdot\ \to\to$ OEIS A003121

70 $\to\leftarrow\to$