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Pattern-avoiding c-Birkhoff polytopes

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Jt. w/

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<u>Main Result</u> We define a "pattern-avoiding" polytope Birk(c) and prove that it is integrally equivalent to the (Stanley's) order polytope $\mathcal{O}(H)$, where H is the heap poset of the c-sorting word of the longest permutation w_0.

a Coxeter elt

Corollary

Volume of Birk(c) = # of linear extensions of H

= # of longest chains in the $\underbrace{\text{c-Cambrian lattice}}$

- its Hasse diagram is an oriented exchange graph of the cluster algebra with initial quiver

  Q = Quiver (c)

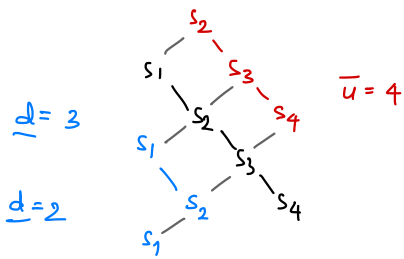
- a Tamari lattice if $c = s_1 s_2 \dots s_n$ OEIS A003121

(a partial order on ways to use parentheses)

I. Setup

Prop The heap H of $\mathrm{sort}_c(w_0)$ is as follows:

(1) Draw a slope -1 "diagonal" D_{long}



(2) Below D_{long}, for each $\underline{d}$,

put a flushed-left diagonal

s_1

$\quad s_2$

$\qquad \ddots$

$\qquad\quad s_{d-1}$

(3) Above D_{long}, for each $\bar{u}$,

put a flushed-right diagonal

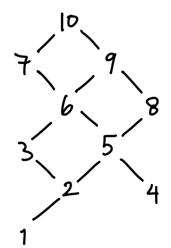
w/ $u-1$ vertices

s_{n-u+2}

$\quad \ddots$

$\qquad s_{n-1}$

$\qquad\quad s_n$

$H =$ 

* The elts of H are $\{1, 2, \dots, \ell\}$, read following the diagonals bottom to top

Connection to rep theory of finite-dimensional algebras

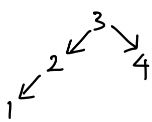
Note:

Rotating H by $90°$ clockwise gives us the "combinatorial AR quiver" for $\mathrm{sort}_c(w_0)$. This terminology is because it is isomorphic to the Auslander–Reiten quiver of Quiver(c).

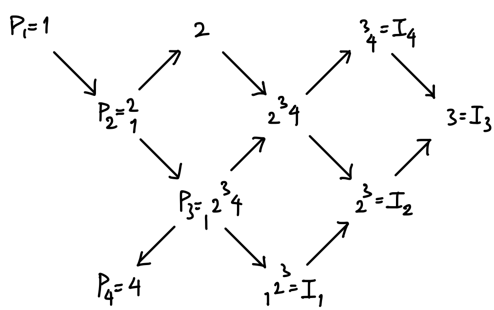
from "Cataland" by Stump, Thomas, Williams

Ex:

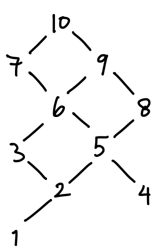
Q = Quiver(c) =



Γ = AR quiver of rep Q =



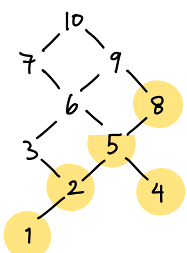
H =



Rotate $90°$

II. The order polytope

Ex:

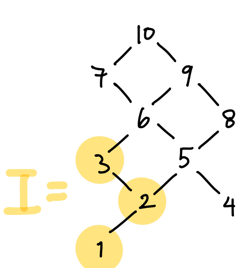
$I =$ 

Indicator vector of $I =$ $\begin{array}{c} 1 \\ 2 \\ \\ 4 \\ 5 \\ \\ \\ 8 \\ \\ \\ \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

III c-singletons

IV. A "pattern-avoiding" Birkhoff subpolytope

Ex:

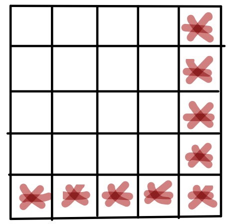


$I = \{1, 2, 3\}$, $\quad w(I) = s_1 s_2 s_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$

$$M(32145) = \begin{array}{|c|c|c|c|c|} \hline & & 3 & & \\ \hline & 2 & & & \\ \hline 1 & & & & \\ \hline & & & 4 & \\ \hline & & & & 5 \\ \hline \end{array} \longrightarrow \text{vector in } \mathbb{R}^{25}$$

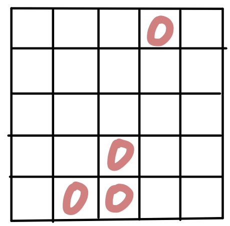
(Birkhoff relation)

- Each row and col sum up to 1:



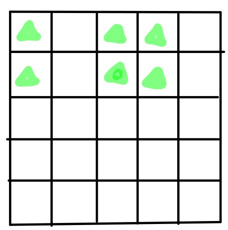
(Zero relation)

- The pattern avoidance conditions immediately require these four entries to be 0:

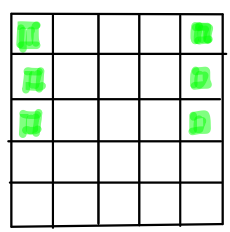


- (Summing relation)

(i) These entries must sum up to 1:



(ii) These entries must sum up to 1:



$\underline{\mathrm{V}}$. Birk$(c) \cong \mathcal{O}(H)$

Ex: Quiver(c) =

$$\begin{tikzcd} & & s_3 \arrow[dl, "\underline{3}"'] \arrow[dr, "\overline{4}"] & \\ & s_2 \arrow[dl, "\underline{2}"'] & & s_4 \\ s_1 & & & \end{tikzcd}$$

$c = (1\ \underline{2}\ \underline{3}\ 5\ \overline{4})$

	$\underline{2}$	$\underline{3}$		
	10	8	0	4
		9	1	5
			2	6
		0		7
3	0	0		

also $\quad \overline{4}$

VI. Inspiration

$n = \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

OEIS A003121 $(1, 1, 1, 2, 12, 286, 33592, \ldots)$ counts:

a. longest chains (length $\binom{n+1}{2}$) in Tamari lattice

(Fishel-Nelson '12)

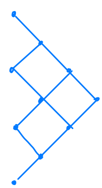
b. (normalized) volume of $B = \mathrm{Conv}($perm. matrices avoiding 312 and 132$)$

(Davis - Sagan '16)

c. linear extensions of poset H whose Hasse diagram is the Auslander - Reiten quiver of $\mathrm{rep}(\leftarrow \leftarrow \cdots \leftarrow)$

(OEIS 3rd comment '03)

ex
$n=4$



d. (normalized) volume of the order polytope $\mathcal{O}(H)$

VII. Questions

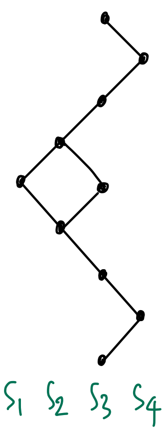
1. Our H is the heap of the c-sorting word of w_0. If we take an arbitrary reduced word $[u]$ of an arbitrary permutation w, and let $H = \mathrm{Heap}([u])$, when is $\mathcal{O}(H) \cong \mathrm{Conv}($perm. matrices of order ideals of $H)$?

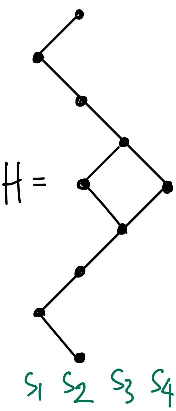
Ans Not in general.

Two counterexamples in A_4:

More counterex in A_5 and bigger A_n

$[u] = [3\ 4\ 3\ 2\ 3\ 1\ 2\ 3\ 4\ 3]$ $\quad$ $[u] = [2\ 1\ 2\ 3\ 2\ 4\ 3\ 2\ 1\ 2]$

$H =$ 

$H =$ 

$\dim \mathcal{O}(H) = 10$, $\dim \mathrm{Birk}(H) = 9$, so $\mathcal{O}(H) \not\cong \mathrm{Birk}(H)$.

2. Relate our results to rep theory meaningfully

3. Generalize to type BDEFGHI ?