A new Motzkin object from box-ball systems

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Solitary waves (solitons)

Scott Russell's first encounter (August 1834)

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped.

[The mass of water in the channel] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, ... and after a chase of one or two miles I lost it in the windings of the channel."

Soliton on the Scott Russell Aqueduct on the Union Canal (July 1995)

(ma.hw.ac.uk/solitons/press.html)

Two soliton animation: www.desmos.com/calculator/86loplpajr

Permutations

Let S_n denote the set of permutations on the numbers $\{1, \ldots, n\}$.

We will represent permutations in *one-line notation*, as

$$w = w(1)\, w(2) \, \cdots \, w(n) \in S_n.$$

Example

A permutation in S_6 in one-line notation: 452361

(Multicolor) box-ball system, Takahashi 1993

A *box-ball system* is a dynamical system of box-ball configurations.

- At each configuration, balls are labeled by numbers 1 through n in an infinite strip of boxes.
- Each box can fit at most one ball.

Example

A possible box-ball configuration:

$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**									$\cdots$

Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.

$t = 0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**				$\cdots$
	$\cdots$	**4**	**5**	**2**	**3**	**6**		**1**			$\cdots$
	$\cdots$	**4**	**5**		**3**	**6**	**2**	**1**			$\cdots$
	$\cdots$	**4**	**5**			**6**	**2**	**1**	**3**		$\cdots$
	$\cdots$		**5**	**4**		**6**	**2**	**1**	**3**		$\cdots$
	$\cdots$			**4**	**5**	**6**	**2**	**1**	**3**		$\cdots$
$t = 1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**	$\cdots$

Box-ball moves ($t = 0$ through $t = 5$)

$t = 0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**															$\cdots$
$t = 1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**												$\cdots$
$t = 2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**									$\cdots$
$t = 3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**						$\cdots$
$t = 4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**			$\cdots$
$t = 5$	$\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future box-ball moves.

Example

The strings **4**, **25**, and **136** are solitons:

$t = 3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**						$\cdots$
$t = 4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**			$\cdots$
$t = 5$	$\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

After a finite number of box-ball moves, the system reaches a *steady state* where:

- each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

Tableaux (English notation)

Definition

- A *tableau* is an arrangement of integers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

1	2	4
3	5	
6	7	

1	3	6
2	5	
4		

1	3	4
2	7	
5	8	
6		

Nonstandard Tableau:

1	2	3
5	6	7
4		

Soliton decomposition

Definition

To construct the *soliton decomposition* $\mathrm{SD}(w)$ of w, start with the one-line notation of w, and run box-ball moves until we reach a steady state; the 1st row of $\mathrm{SD}(w)$ is the rightmost soliton, the 2nd row of $\mathrm{SD}(w)$ is the next rightmost soliton, and so on.

$t=0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**													$\cdots$
$t=1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**										$\cdots$
$t=2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**							$\cdots$
$t=3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**				$\cdots$
$t=4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**	$\cdots$

$\mathrm{SD}(452361) =$

1	3	6
2	5	
4		

with shape $(3, 2, 1)$.

RSK bijection

The classical Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $w = \mathbf{452361}$. Then

$$\mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array} \quad \text{and} \quad \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array} .$$

RSK bijection example

Let $w = \mathbf{452361}$.

$$\text{P}: \quad \mathbf{4} \qquad \begin{array}{cc} 4 & \mathbf{5} \end{array} \qquad \begin{array}{cc} \mathbf{2} & 5 \\ 4 & \end{array} \qquad \begin{array}{cc} 2 & \mathbf{3} \\ 4 & 5 \end{array} \qquad \begin{array}{ccc} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{array} \qquad \begin{array}{ccc} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \qquad \text{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$\text{Q}: \quad \mathbf{1} \qquad \begin{array}{cc} 1 & \mathbf{2} \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \mathbf{3} & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & \mathbf{4} \end{array} \qquad \begin{array}{ccc} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{array} \qquad \text{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Insertion and bumping rule for P

- Insert x into the first row of P.
- If x is larger than every element in the first row, add x to the end of the first row.
- If not, replace the smallest number larger than x in row 1 with x. Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

$Q(w)$ determines the box-ball dynamics of w

Theorem (Cofie–Fugikawa–G.–Stewart–Zeng 2021)

If $Q(v) = Q(w)$, then the soliton decompositions of v and w have the same shape.

Example

$$v = 21435 \text{ and } w = 31425$$

$$Q(v) = Q(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

$$SD(v) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \qquad SD(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

When is SD(w) a standard tableau?

Example

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \quad \mathrm{SD}(21435) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \mathrm{SD}(31425) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Theorem (Drucker–Garcia–G.–Rumbolt–Silver 2020)

Given a permutation w, the following are equivalent:

1. SD(w) is standard
2. SD(w) = P(w)
3. the shape of SD(w) is equal to the shape of P(w)

Definition (good permutations)

A permutation w is *good* if the tableau SD(w) is standard.

Q(w) determines whether w is good

Proposition

Given a standard tableau T, either

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are good,}$$

or

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are not good.}$$

Definition (good tableaux)

A standard tableau T is *good* if $T = \mathrm{Q}(w)$ and w is good.

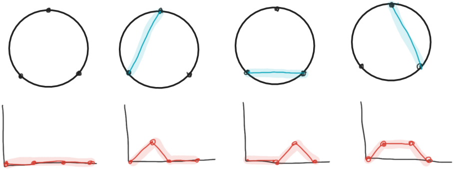
- Question: How many good tableaux are there?

Answer: Good tableaux are Motzkin objects!

Our Main Theorem (G.–Hong–Li–Okonogi-Neth–Sapronov–Stevanovich–Weingord)

The good standard tableaux, $\{Q(w) \mid w \in S_n \text{ and } SD(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

$$M_0 = 1, \qquad M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k \, M_{n-2-k}$$



$M_3 = 4$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

Greene's theorem and localized Greene's theorem

My inspiration and also important tools in our proofs:

- "Greene's theorem" (Greene 1974): The shape of $\mathrm{P}(w)$ records permutation statistics (sizes of largest unions of increasing and decreasing sequences) of w
- "Localized version of Greene's theorem" (Lewis – Lyu – Pylyavskyy – Sen 2019): The shape of $\mathrm{SD}(w)$ records localized versions of these permutation statistics of w

Motzkin recursion and good tableaux

Definition

A sequence of finite sets of objects $A_0, A_1, A_2, \ldots$ is a *Motzkin object* if $|A_0| = |A_1| = 1$, and, for $n \geq 2$,

$$|A_n| = |A_{n-1}| + \sum_{k=0}^{n-2} |A_k| \cdot |A_{n-k-2}|$$

Proof outline

1. Define a goodness-preserving operation on tableaux for every operation in the Motzkin recursion:

   column bump, row wrap, and tilde multiplication

2. Prove that all good tableaux belong to the recursive family generated by these operations

Tilde product

We define a slight variation of tableaux multiplication (from Fulton's "Young tableaux") and call it *tilde multiplication*.

Example

Consider $T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$, $T_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. Define $\overline{T_1} = \begin{array}{|c|c|c|}\hline 3 & 4 & 7 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$. Then we compute that

$$T_1 \widetilde{\times} T_2 = \overline{T_1} \times T_2 = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Theorem (Tilde multiplication preserves goodness)

Suppose T_1, T_2 are standard tableaux. Then, $T_1 \widetilde{\times} T_2$ is good if and only if T_1 and T_2 are good.

Column bump

The *column bump* of T is $\mathrm{bump}(T) = T \,\widetilde{\times}\, \boxed{1}$.

Example

$$\text{Take } T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}, \; T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}.$$

$$\text{Then } \mathrm{bump}(T_1) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}, \; \mathrm{bump}(T_2) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Theorem (Bump preserves goodness)

T is good if and only if bump(T) is good.

Row wrap

Let T be a standard tableau of size n. The *row wrap* of T, denoted $\mathrm{wrap}(T)$, is constructed as follows:

- Increase every element of T by 1 to get T'
- Prepend the first row of T' by 1
- Append the first row of T' by $n+2$

Example

$$\text{Take } T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}, \; T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}.$$

$$\text{Then } \mathrm{wrap}(T_1) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 8 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}, \; \mathrm{wrap}(T_2) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

Theorem (Wrap preserves goodness)

T is good if and only if wrap(T) is good.

Recursive family of good tableaux

Definition

Let $K_0 = \{\emptyset\}$ and $K_1 = \left\{\begin{array}{|c|}\hline 1 \\ \hline\end{array}\right\}$. Recursively define a set K_n of size-n good tableaux:

- for each $Q \in K_{n-1}$,

$$\text{bump}(Q) \in K_n$$

- for each pair of $Q_1 \in K_k$ and $Q_2 \in K_{n-k-2}$ for $0 \le k \le n-2$,

$$Q_1 \widetilde{\times} \text{wrap}(Q_2) \in K_n$$

Example

$$K_2 = \left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right\}, \quad K_3 = \left\{\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right\}$$

K_n as a Motzkin object

Recall the Motzkin recursion.

Definition

A sequence of finite sets of objects $A_0, A_1, A_2, \ldots$ indexed by n is said to be a *Motzkin object* if $|A_0| = |A_1| = 1$ and for $n \geq 2$,

$$|A_n| = |A_{n-1}| + \sum_{k=0}^{n-2} |A_k| \cdot |A_{n-k-2}|$$

Theorem

- *The K_n tableaux are a Motzkin object.*
- *A tableau T of size n is good if and only if $T \in K_n$. In particular, good tableaux of size n are a Motzkin object.*

Further question: Characterize good permutations using consecutive permutation patterns.

- A corollary of our work + Elizalde et al is that good permutations can be characterized by consecutive pattern avoidance.
- Note: Good permutations are impossible to classify using classical permutation patterns.
- Note (added after the talk): Elizalde informed us that the required set of consecutive permutation patterns is infinite

Further question: Is there a natural way to generalize the tilde product to (non-standard) soliton decomposition tableaux?

- Given two standard tableaux T_1 and T_2, the tilde product $T_1 \widetilde{\times} T_2$ is constructed simply by starting with T_2 and adding columns below it.

- In our work, we prove that the box-ball soliton partition $\Lambda(T_1 \widetilde{\times} T_2)$ of $T_1 \widetilde{\times} T_2$ can be constructed by starting with the box-ball soliton partition $\Lambda(T_2)$ of T_2 and then adding columns of $\Lambda(T_1)$ below it.

  This makes us think that we should be able to define a similar tilde product directly on (non-standard) soliton decomposition tableaux.

Further question: Is there a natural subclass of the good tableaux which is enumerated by the Catalan numbers?

Y	O	U	!
A	N	K	
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Open Problem Session 3-6pm today in Room 226

You're welcome to join us for the Open Problem Session (of the "Geometric and Algebraic Combinatorics" session).

- When: 3pm today
- Location: 226, Hartford Times Building
- There is a whiteboard so you can present on the board (or project a tablet or slides)
- Target audience: grad students and recent PhDs
- In addition to listening, you are welcome to present problem(s), 10–15 mins per problem.
- Please email me at emily_gunawan@UML.edu if you wish to present.

Greene's theorem, slide 1/3

Definition (longest k-increasing subsequences)

A subsequence σ of w is called *k-increasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an increasing subsequence of w. Let $\mathrm{i}_k := \mathrm{i}_k(w)$ denote the length of a longest k-increasing subsequence of w.

Example (Let $w = 5623714$.)

- The longest 1-increasing subsequences are 567, 237, and 234.
- The longest 2-increasing subsequence is given by $562374 = 567 \sqcup 234$.
- A longest 3-increasing subsequence (among others) is given by $5623714 = 56 \sqcup 237 \sqcup 14$.
- Thus, $\mathrm{i}_1 = 3$, $\mathrm{i}_2 = 6$, and $\mathrm{i}_k = 7$ if $k \geq 3$.

Greene's theorem, slide 2/3

Definition (longest k-decreasing subsequences)

Similarly, a subsequence σ of w is called *k-decreasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an decreasing subsequence of w. Let $\mathrm{d}_k := \mathrm{d}_k(w)$ denote the length of a longest k-decreasing subsequence of w.

Example (Let $w = 5623714$.)

- The longest 1-decreasing subsequences are
  521, 621, 531, and 631.
- A longest 2-decreasing subsequence (among others) is given by $52714 = 521 \sqcup 74$.
- A longest 3-decreasing subsequence (among others) is given by $5623714 = 52 \sqcup 631 \sqcup 74$.
- Thus, $\mathrm{d}_1 = 3$, $\mathrm{d}_2 = 5$, and $\mathrm{d}_k = 7$ if $k \geq 3$.

Greene's theorem, slide 3/3

Theorem (Greene, 1974)

Suppose $w \in S_n$*. Let* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ *denote the RS partition of* w*, that is, let* $\lambda = \operatorname{sh} P(w)$*. Let* $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ *denote the conjugate of* λ*. Then, for any* k*,*

$$\mathrm{i}_k(w) = \lambda_1 + \lambda_2 + \ldots + \lambda_k,$$
$$\mathrm{d}_k(w) = \mu_1 + \mu_2 + \ldots + \mu_k.$$

Example

By Greene's theorem, the RS partition is equal to $\lambda = (\mathrm{i}_1, \mathrm{i}_2 - \mathrm{i}_1, \mathrm{i}_3 - \mathrm{i}_2) = (3, 3, 1)$. We can verify this by computing the RS tableaux

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}, \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

A localized version of Greene's theorem, slide 1/3

Definition (A localized version of longest k-increasing subsequences)

Let $\mathrm{i}(u) :=$ the length of a longest increasing subsequence of u.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{I}_k(w) = \max_{w=u_1|\cdots|u_k} \sum_{j=1}^{k} \mathrm{i}(u_j)$, where the maximum is taken over ways of writing w as a concatenation $u_1 \mid \cdots \mid u_k$ of consecutive subsequences.

Example

Let $w = 5623714$. For short, we write $\mathrm{I}_k := \mathrm{I}_k(w)$. Then

$\mathrm{I}_1 = \mathrm{i}(w) = 3$ (since the longest increasing subsequences are 567, 237, 234),

$\mathrm{I}_2 = 5$ (witnessed by $56|23714$ or $56237|14$),

$\mathrm{I}_3 = 7$ (witnessed uniquely by $56|237|14$), and

$\mathrm{I}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem, slide 2/3

Definition (A localized version of longest k-decreasing subsequences)

Let $\mathrm{D}(u) \coloneqq 1 + |\{\text{descents of } u\}|$.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{D}_k(w) = \max_{w = u_1 \sqcup \cdots \sqcup u_k} \sum_{j=1}^{k} \mathrm{D}(u_j)$, where the maximum is taken over ways to write w as the union of disjoint subsequences u_j of w.

Example

Let $w = 5623714$. For short, we write $\mathrm{D}_k \coloneqq \mathrm{D}_k(w)$. Then

$\mathrm{D}_1 = \mathrm{D}(w) = 1 + |\text{descents of } 5623714| = 1 + |\{2, 5\}| = 3$,

$\mathrm{D}_2 = 6$ (take subsequences 531 and 6274, among other partitions),

$\mathrm{D}_3 = 7$ (take subsequences 52, 631, and 74, among other partitions), and

$\mathrm{D}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem, slide 3/3

Theorem (Lewis–Lyu–Pylyavskyy–Sen 2019)

Suppose $w \in S_n$. *Let* $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \dots)$ *denote* $\mathrm{sh}\, SD(w)$. *Let* $M = (M_1, M_2, M_3, \dots)$ *denote the conjugate of* Λ. *Then, for any* k,

$$\mathrm{I}_k(w) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_k,$$
$$\mathrm{D}_k(w) = M_1 + M_2 + \ldots + M_k.$$

Example

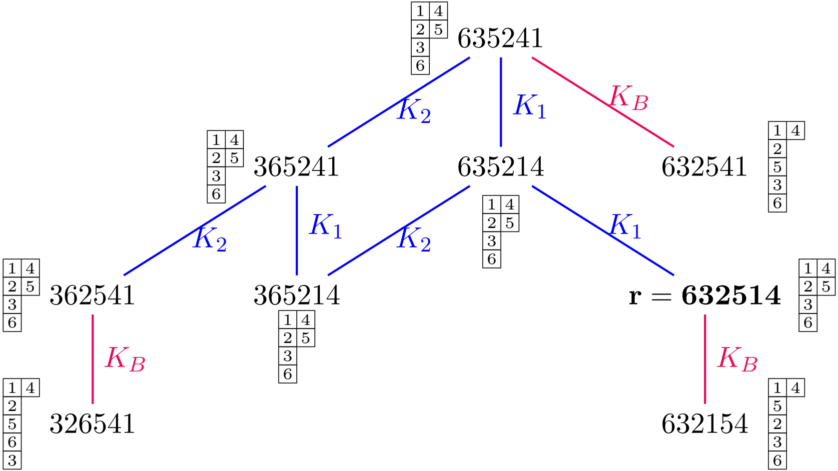
Let $w = 5623714$. By the above theorem, $\mathrm{sh}\,\mathrm{SD}(w) = (\mathrm{I}_1, \mathrm{I}_2 - \mathrm{I}_1, \mathrm{I}_3 - \mathrm{I}_2) = (3, 2, 2)$. We can verify this by computing the soliton decomposition $\mathrm{SD}(w)$, which turns out to be the (non-standard) tableau

1	3	4
2	7	
5	6	

.

Note: $\mathrm{sh}\,\mathrm{SD}(w) = (3, 2, 2)$ is smaller than $\mathrm{sh}\, P(w) = (3, 3, 1)$ in the dominance order.

Further question: Characterize permutations with the same soliton decomposition



Permutations connected by *Knuth moves* to $\mathbf{r} = \mathbf{632514}$ and their soliton decompositions