# Snake Graphs

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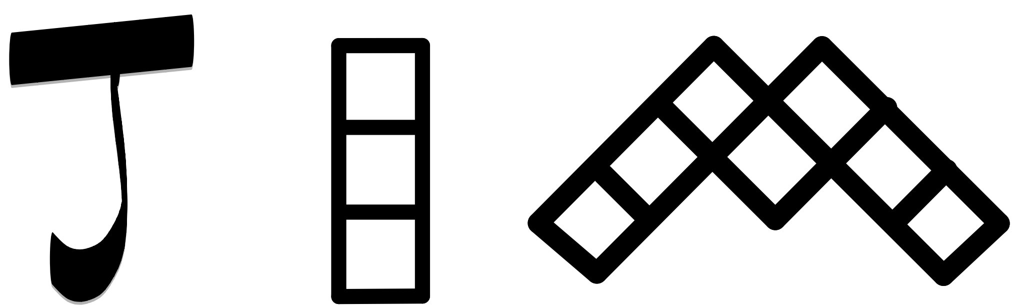
Statistical & Dynamical Combinatorics



Wed, June 26 2024

Jim's 2005 article based on work w/ D. Thurston &
Boston-area undergraduates starting in 2001

Mathematics > Combinatorics

*[Submitted on 25 Nov 2005 (v1), last revised 28 May 2020 (this version, v5)]*

# The combinatorics of frieze patterns and Markoff numbers

James Propp

This article, based on joint work with Gabriel Carroll, Andy Itsara, Ian Le, Gregg Musiker, Gregory Price, Dylan Thurston, and Rui Viana, presents a combinatorial model based on perfect matchings that explains the symmetries of the numerical arrays that Conway and Coxeter dubbed frieze patterns. This matchings model is a combinatorial interpretation of Fomin and Zelevinsky's cluster algebras of type A. One can derive from the matchings model an enumerative meaning for the Markoff numbers, and prove that the associated Laurent polynomials have positive coefficients as was conjectured (much more generally) by Fomin and Zelevinsky. Most of this research was conducted under the auspices of REACH (Research Experiences in Algebraic Combinatorics at Harvard).

Comments: Presented at the 18th International Conference on Formal Power Series and Algebraic Combinatorics. (Revised June 2006: I corrected a mis-statement at the end of section 2, and added reference to recent unpublished work of Hickerson. Revised May 2007: I correct a typo and added a paragraph.) Published as Integers, Volume 20 (2020), article A12; http://math.colgate.edu/~integers/u12/u12.pdf

Subjects: **Combinatorics (math.CO)**

## This talk

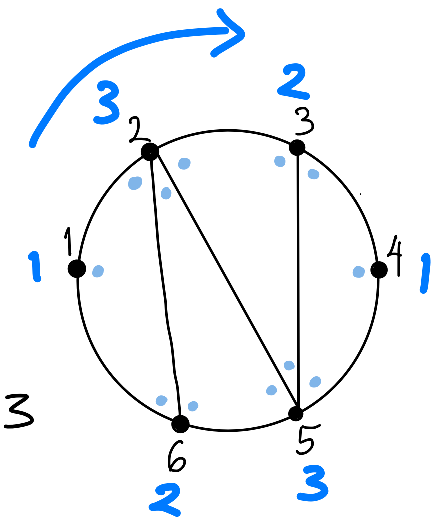
- Conway-Coxeter friezes
- cluster algebras of type A
- Several versions of "snakes" from Jim's paper
- Subwords of binary numbers

# Conway–Coxeter friezes ($A_n$ friezes), 1970s

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | |
| | 1 | | 3 | | 2 | | 1 | | 3 | | 2 | | 1 |
| 1 | | 2 | | 5 | | 1 | | 2 | | 5 | | 1 | |
| | 1 | | 3 | | 2 | | 1 | | 3 | | 2 | | 1 |
| 1 | | 1 | | 1 | | 1 | | 1 | | 1 | | 1 | |

quiddity sequence

$n = 3$

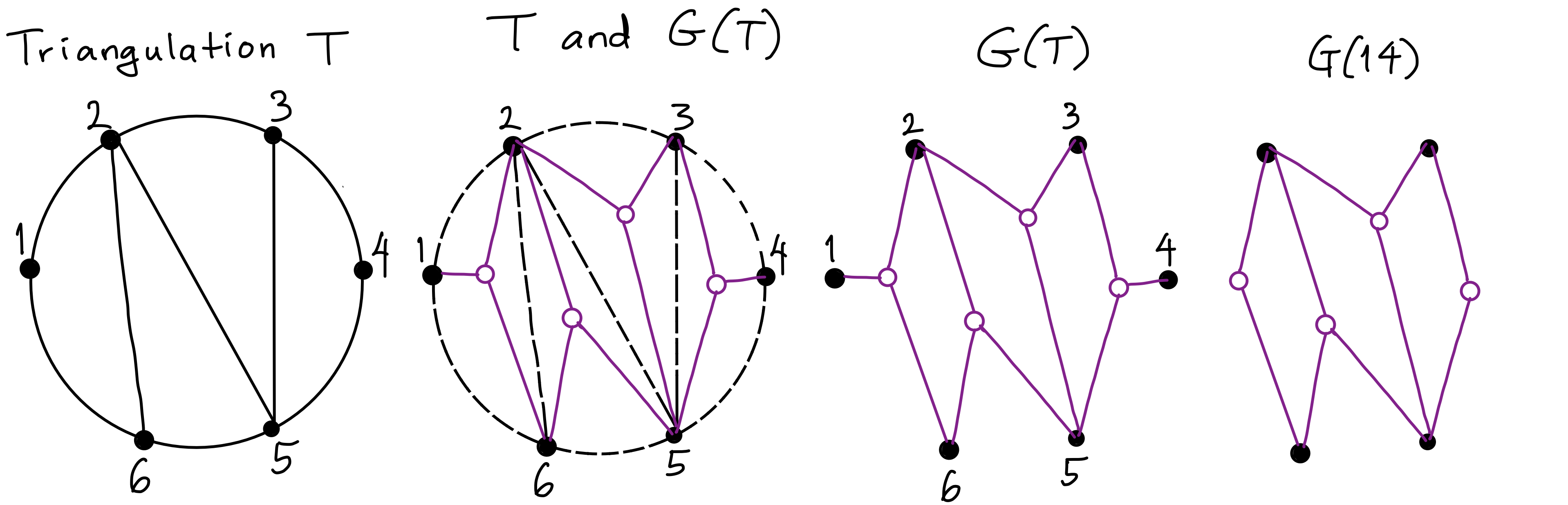



$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \rightsquigarrow ad - bc = 1$$

[Conway – Coxeter 1970s]

- $A_n$ friezes $\longleftrightarrow$ triangulations of $(n+3)$-gon
- The array is invariant under a glide reflection

## A bipartite dual graph $G(T)$ of a triangulation $T$ and $G(ij)$ for each diagonal $(ij)$.



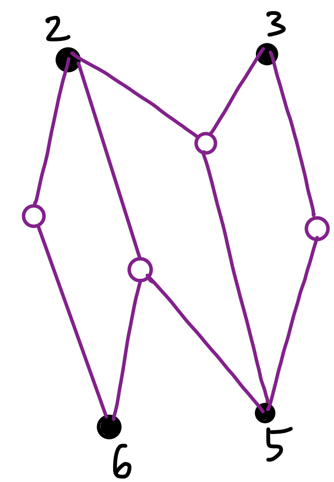

- black vertices ● of $G$ $\leftrightarrow$ vertices of the triangulation $T$
- white vertices ○ of $G$ $\leftrightarrow$ triangular faces of $T$
- edges of $G(T)$ $\leftrightarrow$ all incidences between vertices & faces in $T$

Note : $\underbrace{\#\text{ black vertices}}_{b:=} = 2 + \#\text{ white vertices}$

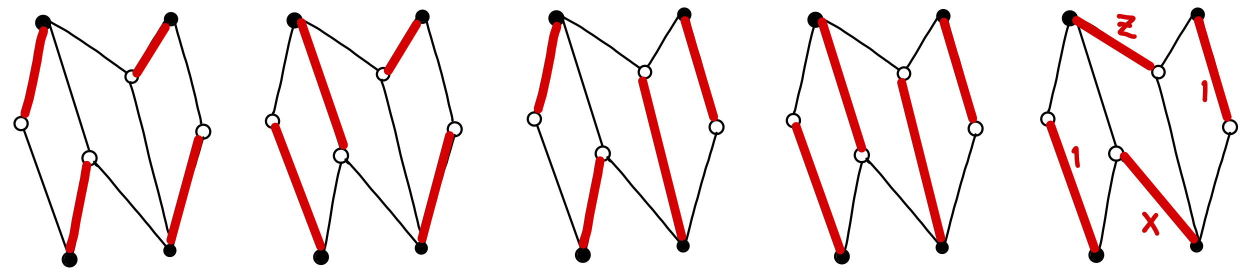
For $i \neq j$ in $\{1, \ldots, b\}$, $G(ij) := G(T) - \{\text{vertices } i \text{ and } j\}$

**Def** $m(ij) = \#$ perfect matchings of $G(ij)$

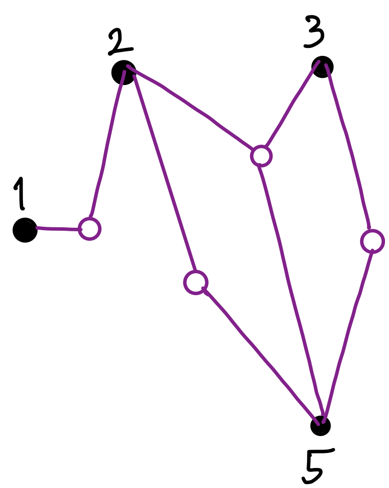
$G(14)$



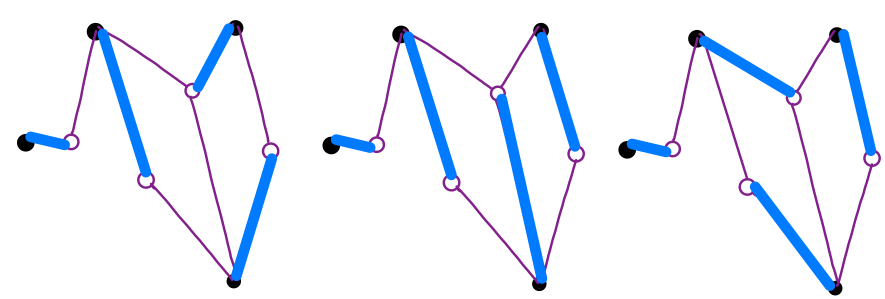

$m(14) = 5$




$G(46)$




$m(46) = 3$



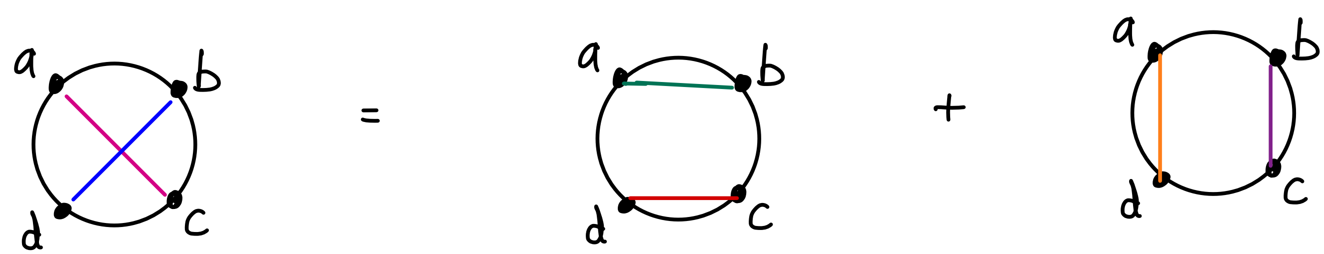
# Kuo's condensation lemma ['03]

If $G$ is a bipartite graph w/

# black vertices = 2 + # white vertices and

$a, b, c, d$ are black vertices around a face of $G$, then

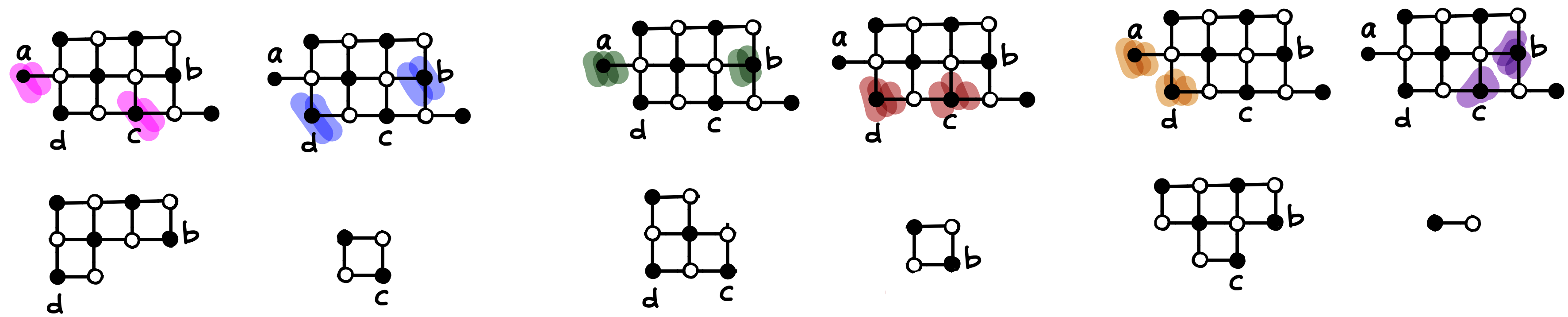
$$m(ac)\, m(bd) = m(ab)\, m(cd) + m(ad)\, m(bc)$$




Ex:

G(ij) :

Remove forced & forbidden edges :




$$7 \times 2 = 4 \times 2 + 6 \times 1$$

Thm (Carrol – Price 2003, which built on Itsara-Le-Musiker-Price-Viana)

The Conway-Coxeter frieze of a triangulation is equal to

$$\begin{matrix} \cdots & m(12) & & m(23) & & m(34) & & m(45) & & \cdots \\ \cdots & & m(13) & & m(24) & & m(35) & & & \cdots \\ \cdots & m(n3) & & m(14) & & m(25) & & m(36) & & \cdots \\ \cdots & & m(n4) & & m(15) & & m(26) & & & \cdots \\ & \vdots & & \vdots & & \vdots & & \vdots & & \end{matrix}$$

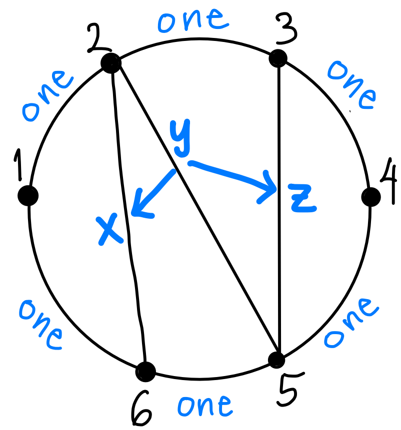
(Interpret subscripts mod n)

This explains the glide-symmetry of a frieze,

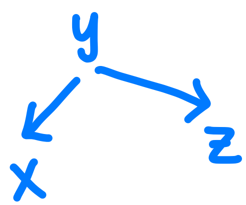
since $G(ij) = G(ji)$

[Fomin–Zelevinsky '01–'02]

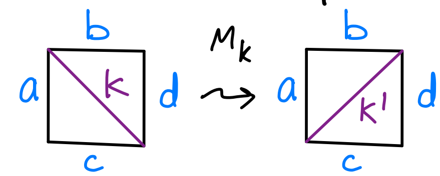
## from a triangulation to type $A_n$ cluster variables




Quiver $Q(T)$:



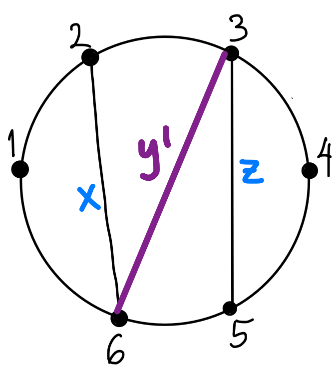

Mutation / flip rule:



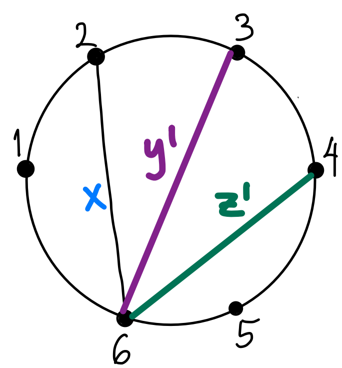

$$k' = \frac{ad + bc}{k}$$

- Put weight along edges: Boundary edges $M(i, i+1) := 1$
  $M(26) = x$, $M(25) := y$, $M(35) := z$
- The rest of diagonals $M(ij)$ are rational functions computed recursively following the mutation rule

Ex: $M(36) = y' = \dfrac{xz + 1}{y}$

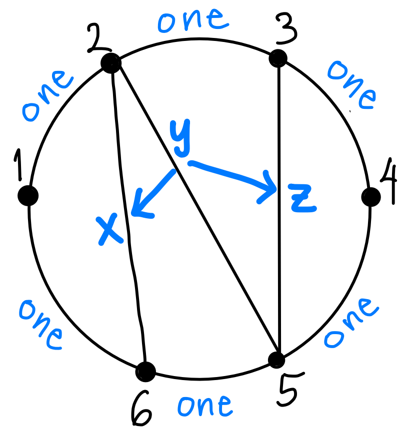



$$M(46) = z' = \frac{y' + 1}{z} = \frac{\left(\frac{xz + 1}{y}\right) + 1}{z} = \frac{xz + 1 + y}{yz}$$

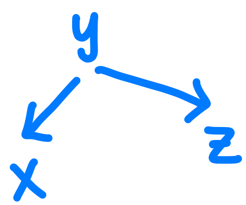


[Fomin-Zelevinsky '01-'02]

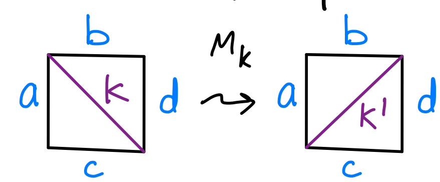
## from a triangulation to type $A_n$ cluster variables




Quiver $Q(T)$:




Mutation / flip rule:




$$k' = \frac{ad+bc}{k}$$

- Put weight along edges: Boundary edges $M(i,i+1) := 1$
  $M(26) = x$, $M(25) := y$, $M(35) := z$
- The rest of diagonals $M(ij)$ are rational functions computed recursively following the mutation rule

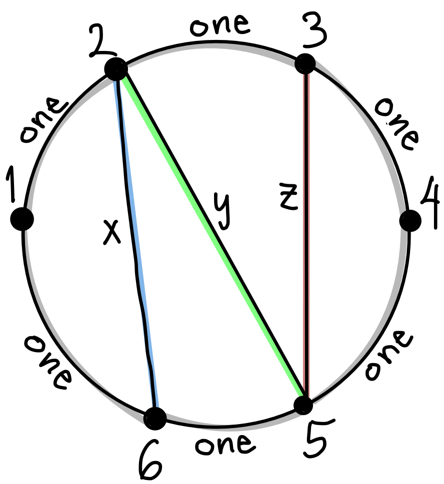
- cluster variables: All rational functions $M(ij)$

- [Laurent Phenomenon 2001]
  In general, the cluster variables are Laurent polynomials

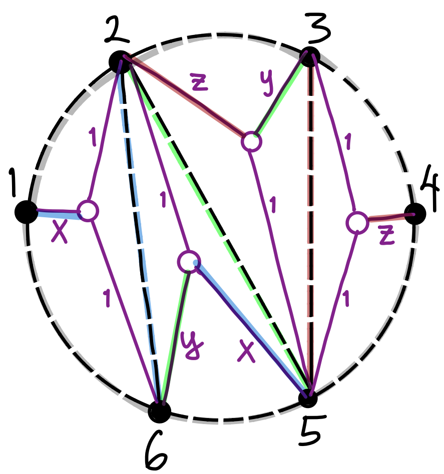
# The original perfect matching formula [Propp et al '05]

$$M(ij) = \left[ \sum_{\substack{\text{perfect} \\ \text{matchings} \\ pm}} \text{weight of } pm \text{ of } G(ij) \right] \Big/ \left( \text{weight of all diagonals} \right)$$

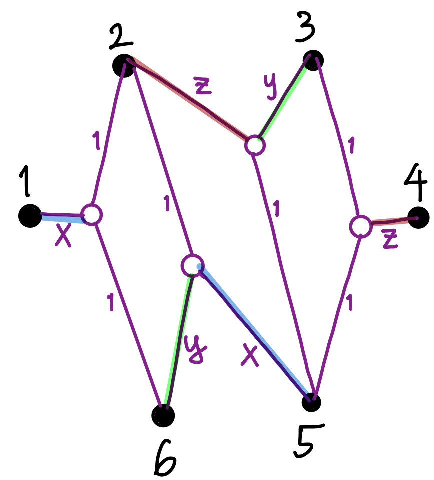
Triangulation T



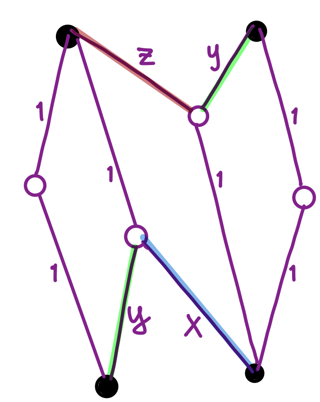

T and G(T)

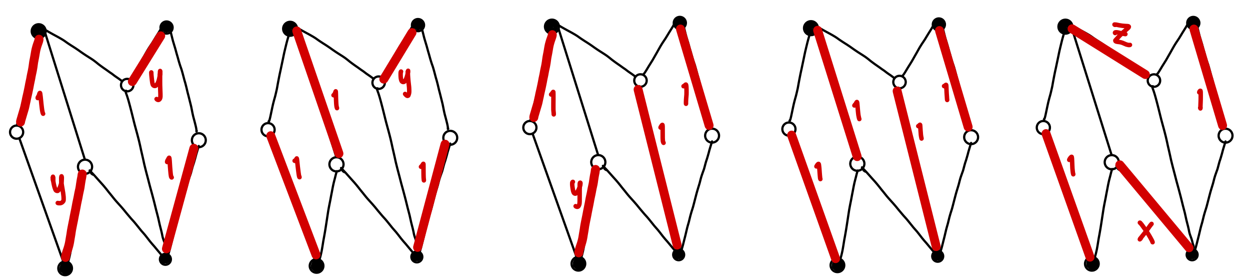



G(T)




G(14)




$$M(14) = \left[ \; y^2 + y + y + 1 + xz \; \right] \Big/ (xyz)$$

**Positivity "Conjecture" of cluster variables** (proven in general in '13, '14):

All coefficients of each Laurent polynomial are positive

**Corollary of the original perfect matching formula:**

In type A, the cluster variables are Laurent polynomials w/ positive coefficients

**Thm** (Propp et al '05)

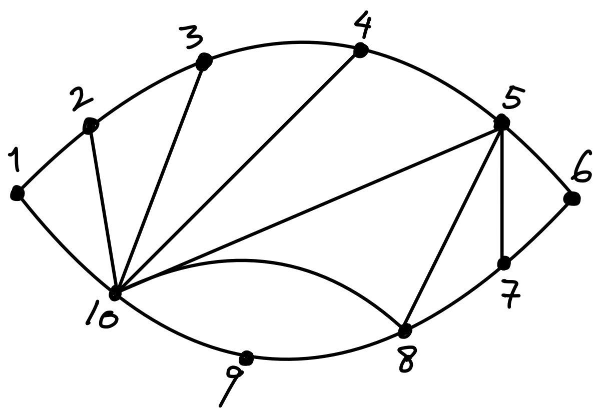
The cluster variables for type A form a Conway-Coxeter frieze w/ rational functions as entries

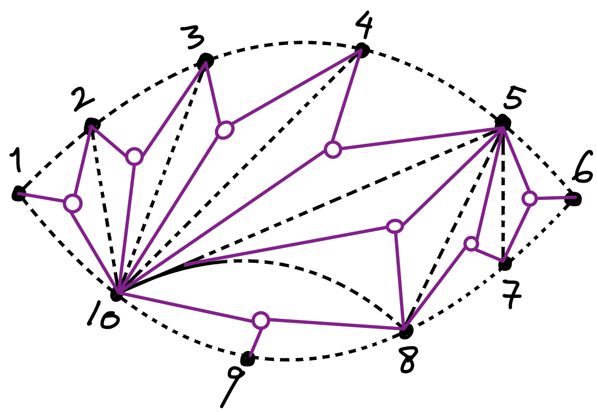
$$\begin{array}{ccccccccccc}
\cdots & M(12) & & M(23) & & M(34) & & M(45) & & \cdots \\
\cdots & & M(13) & & M(24) & & M(35) & & & \cdots \\
\cdots & M(n3) & & M(14) & & M(25) & & M(36) & & \cdots \\
\cdots & & M(n4) & & M(15) & & M(26) & & & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(Interpret subscripts mod n)

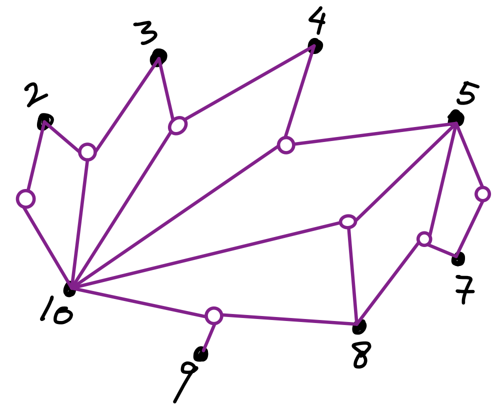
# From bipartite graph $G(ij)$ to many versions of "snakes"

more convenient than $G(ij)$ for computation



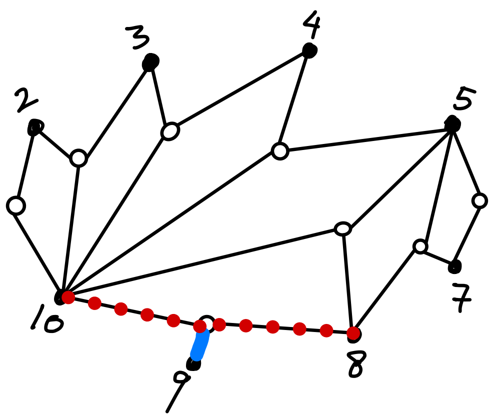




$G(16)$

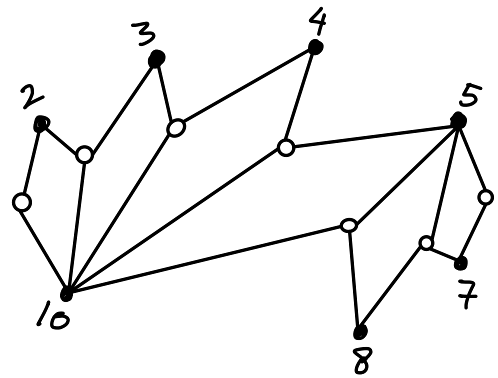



Shear forced edges ( belonging to every pm)

and forbidden edges ( cannot be in a pm)

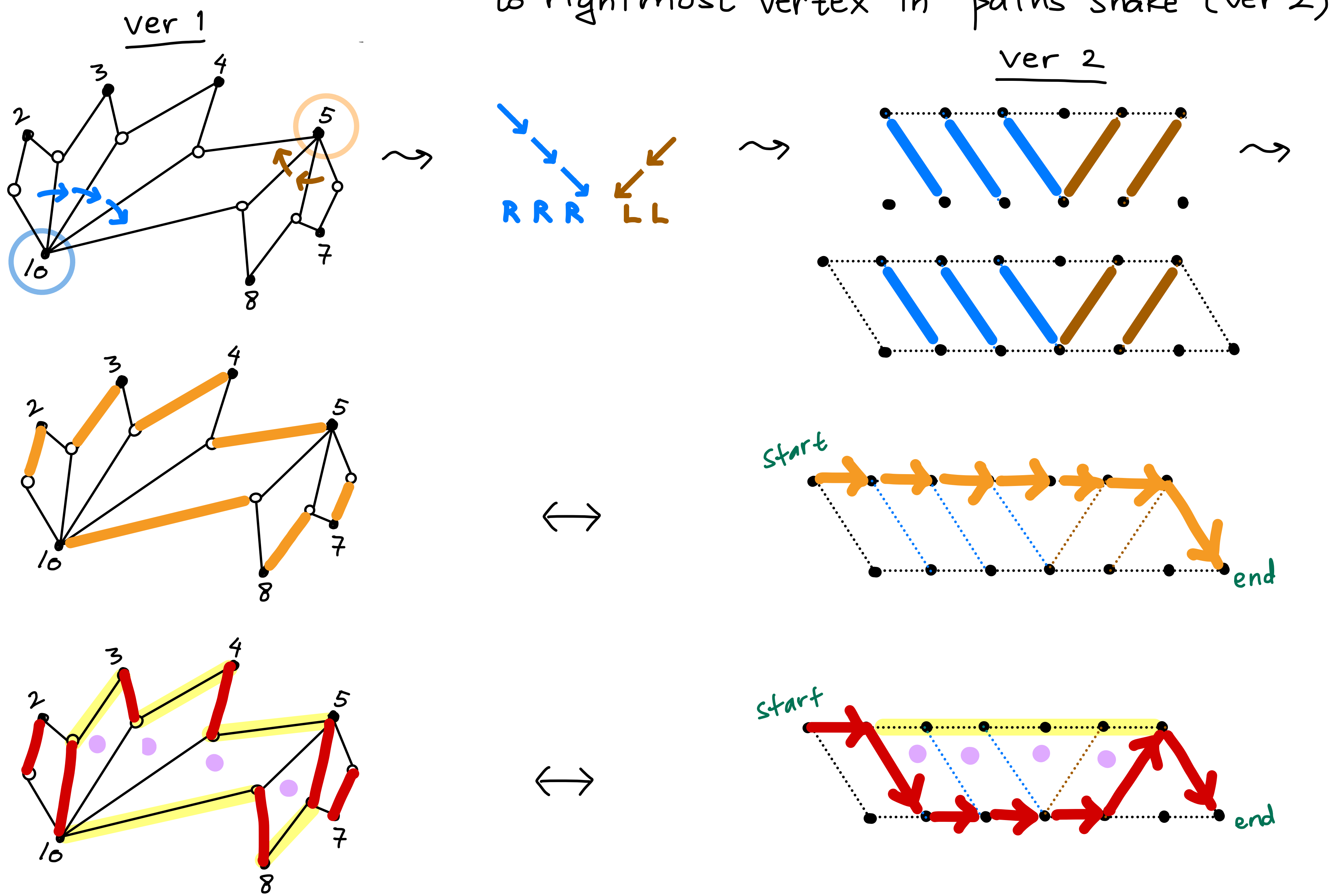



shear $\rightsquigarrow$



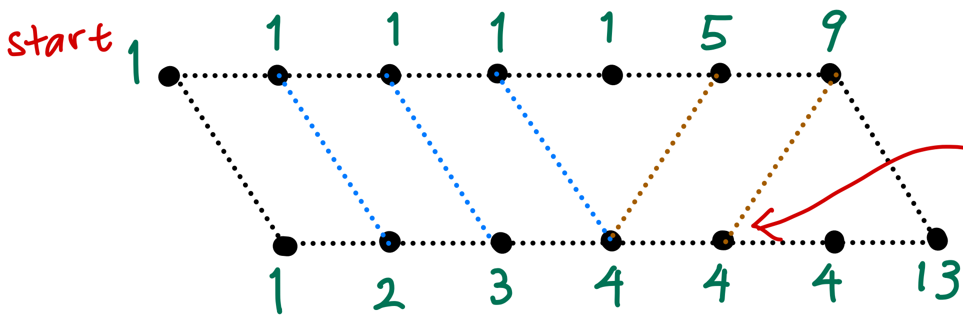

# pm's of $G(16)$ = # pm's of this "snake" (version 1)

# pm's of "snake" (ver 1) = # forward paths from leftmost vertex to rightmost vertex in "paths snake" (ver 2)



# Conway-Coxeter's "primary school algorithm"

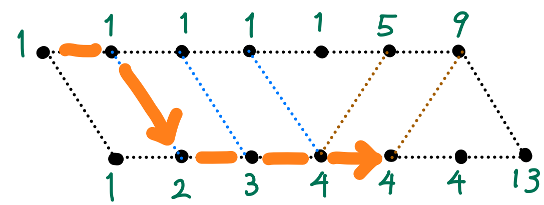
also works for "paths snake" (ver 2)



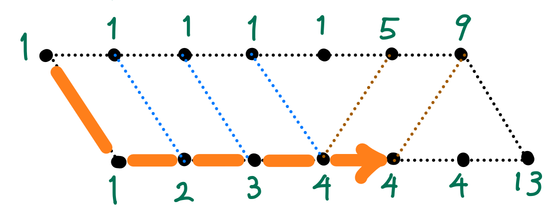

# of paths from "start" to this vertex = 4

The four paths:

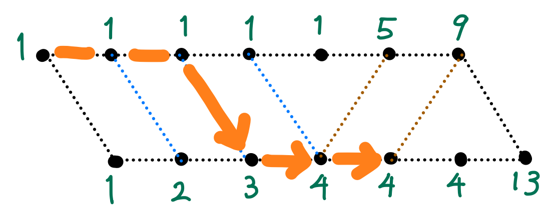
1.



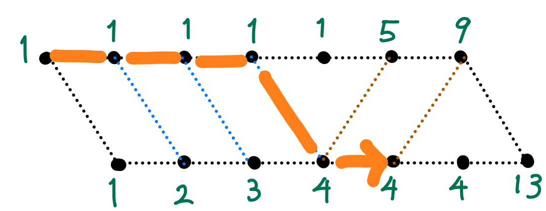

2.



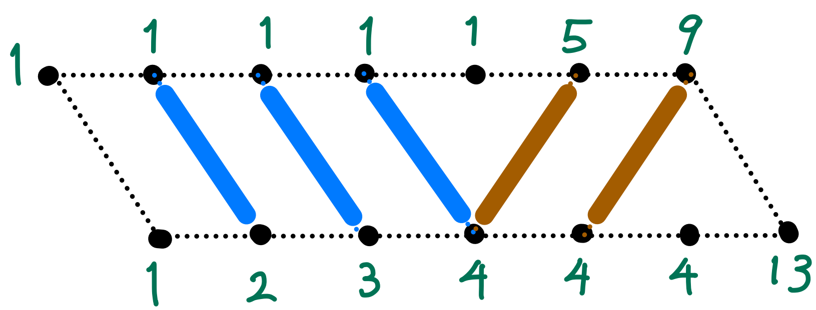

3.

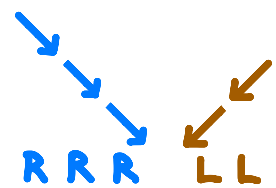


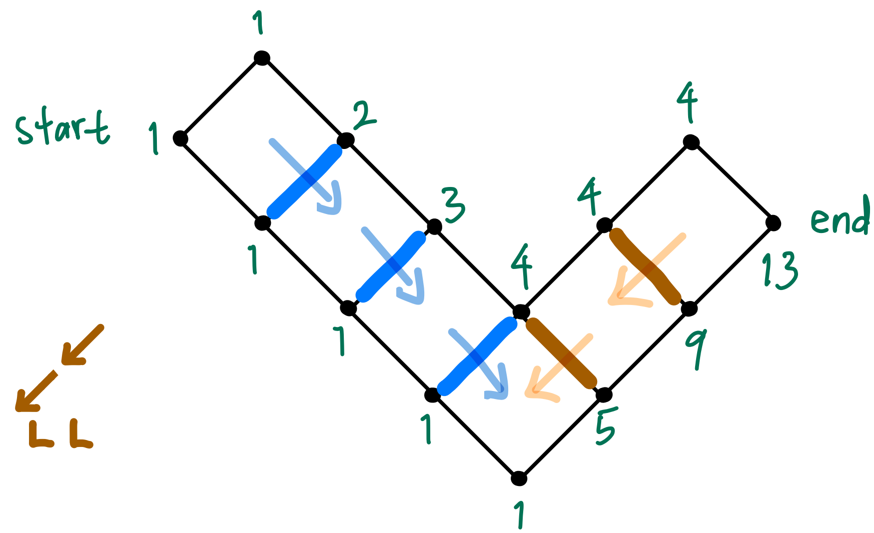

4.



# "paths snake" (ver 2) as a border strip skew Young diagram



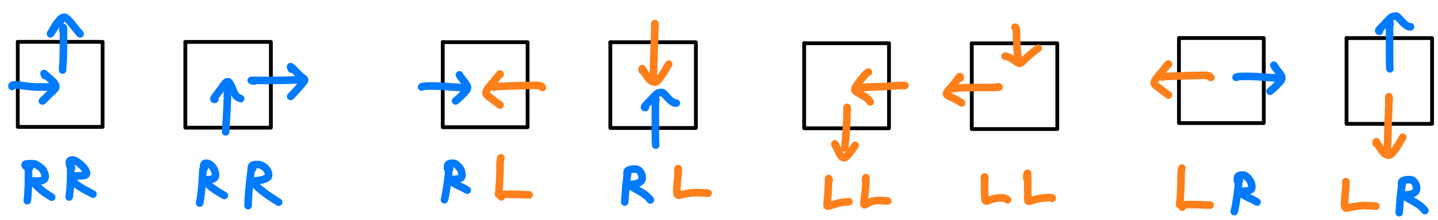



# The snake graph (ver 3)

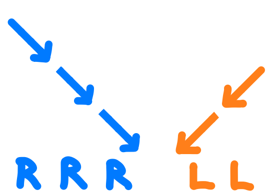
- a dual of the "paths snake" (ver 2)
- used most frequently
- # paths from left to right in "paths snake" (ver 2)
  = # perfect matchings of the snake graph

Given quiver, start w/ a square □,

build a graph by gluing square east/north of the form

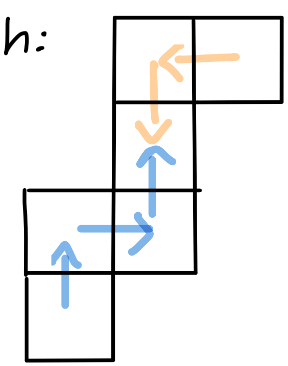



Ex: Quiver:

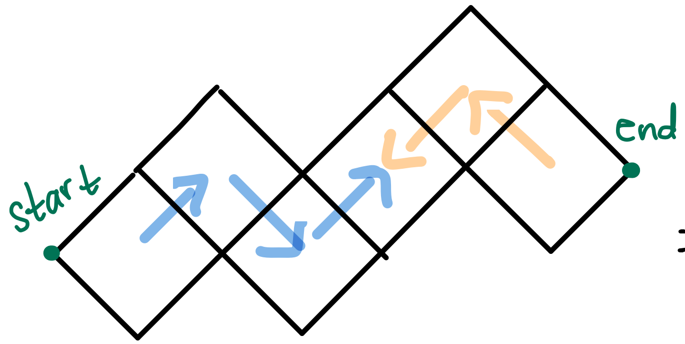



$\rightsquigarrow$

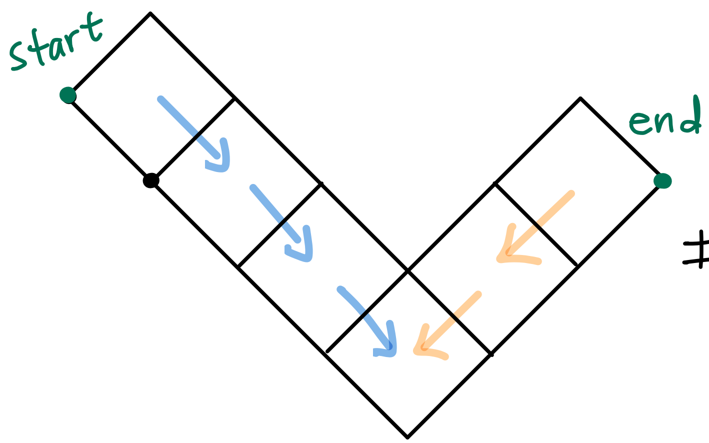
The snake graph:
(ver 3)



# Duality of the snake graph (ver 3) and the "paths snake" (ver 2):



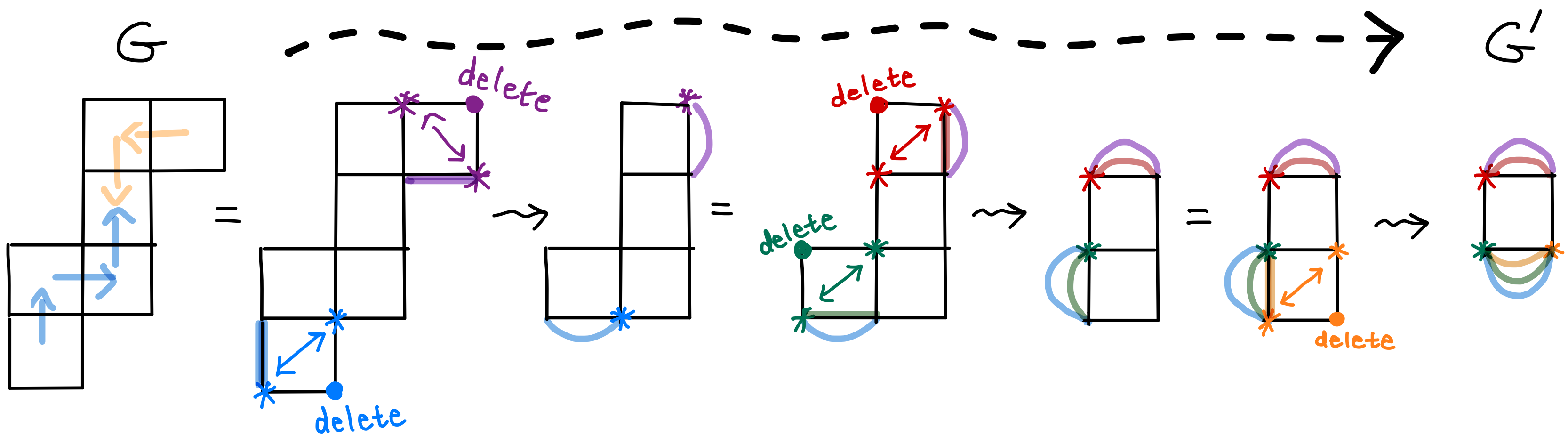

# perfect matchings = 13, # paths = 19




# perfect matchings = 19, # paths = 13

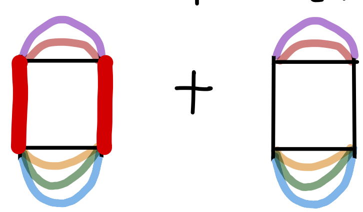
# The snake graph with multiple edges (ver 4)

- Turn snake graph $G$ into a straight <u>snake multigraph</u> by




- A "folklore of perfect matchings": #pm's of $G$ = #pm's of $G'$

Ex: # pm's is 1 + 3(4) = 13



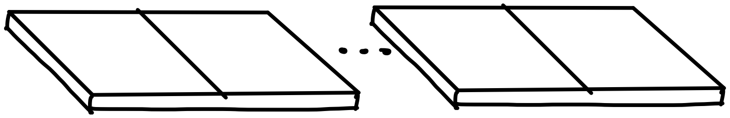
matchings w/ $\underbrace{\text{north}}_{3 \text{ options}}$ and $\underbrace{\text{south}}_{4 \text{ options}}$ edges

- In general #(max sequences of $\underset{R}{\rightarrow} \underset{R}{\rightarrow} \cdots \underset{R}{\rightarrow}$ and $\underset{L}{\leftarrow} \underset{L}{\leftarrow} \cdots \underset{L}{\leftarrow}$)

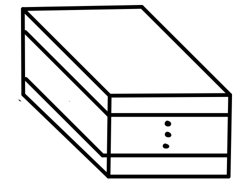
  $= 1 + \# \square$s in the snake multigraph (ver 4)

[Benjamin – Quinn '03 "Proofs that really count"]

Tilings of a board by "dominoes" and "stackable squares"

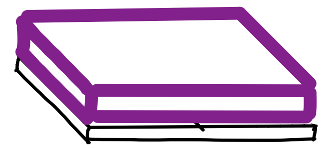
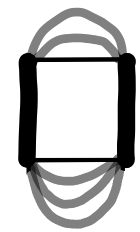


$1 \times 2$

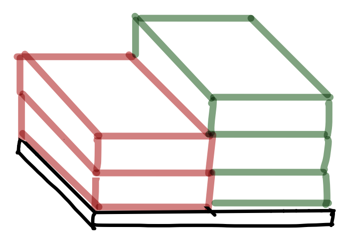
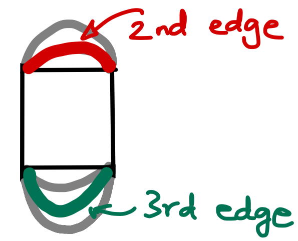


$\xleftrightarrow{\text{bij}}$ the pm's of the snake multigraph (ver 4)

- Here, the board is $1 \times 2$ & the possible tilings are
  - one possibility: one domino tile

 $\leftrightarrow$ 

  - 12 possibilities:
    - put 1, 2, or 3 square tiles on the first square; and
    - put 1, 2, 3, or 4 square tiles on the second square

 $\leftrightarrow$ 

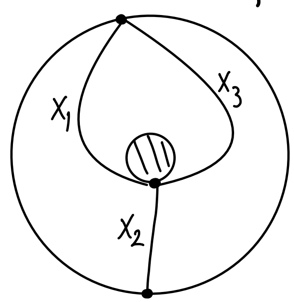

- # of tilings can be computed using continued fraction

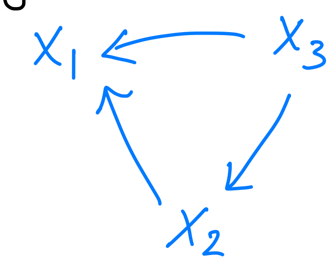
ex $[3,4] = 3 + \frac{1}{4} = \frac{13}{4}$ ← # of tilings

# Snake graphs since 2005

- [Musiker – Propp '06] used similar models to prove positivity of cluster variables of rank 2 affine type

- [Fomin – Shapiro – Thurston '06, Cluster algebras from triangulations of general marked surfaces]




called type $\tilde{A}_{12}$

one arrow counterclockwise

arrows clockwise

→ [Musiker – Schiffler – Williams '09 – '10]

The snake graphs (ver 3) were used to prove positivity of all cluster algebras from surfaces & to study bases

- [Çanakçı – Schiffler '12 – '17]

Abstract snake graphs & continued fractions

⋮ ⋮

- Researchers interested in cluster theory & combinatorial models

[Bailey – G. '18 – '19] Perfect matchings of a snake graph $\overset{\text{bij}}{\longleftrightarrow}$ subwords of base-2 expansions of a natural number

<u>Def</u> The binomial coefficient $\binom{w}{v}$ of two words is the # of times $v$ occurs as a subsequence of $w$

Ex: $w = 101001$ $\qquad \binom{w}{v} = 6$

$v = 101$

1 0 1 0 0 1    1 0 1 0 0 1    1 0 1 0 0 1    1 0 1 0 0 1    1 0 1 0 0 1    1 0 1 0 0 1

<u>Def</u> • If $\binom{w}{v} \geqslant 1$, we say $v$ is a <u>subword</u> of $w$.

• Given a positive integer, we can write its binary expansion as a binary word $w = 1 w_2 w_3 \cdots w_\ell$.

Let $Q(w)$ be the type $A_\ell$ quiver w/ vertices $1, \dots, \ell$ st:

$i-1 \xrightarrow{0} i$ if $w_i = 0$ and $i-1 \xleftarrow{1} i$ if $w_i = 1$

Ex: $Q(101001)$

$1 \xrightarrow{0} 2 \xleftarrow{1} 3 \xrightarrow{0} 4 \xrightarrow{0} 5 \xleftarrow{1} 6$

[Bailey – G. '18 – '19] Perfect matchings of a snake graph $\xleftrightarrow{\text{bij}}$ subwords of base-2 expansions of a natural number

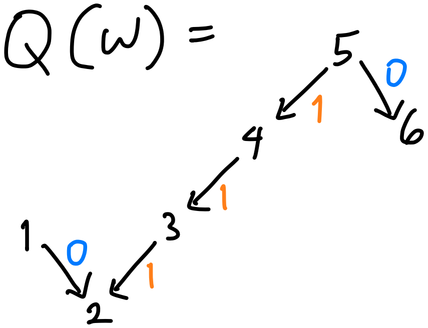
[Bailey – G. '18 – '19]

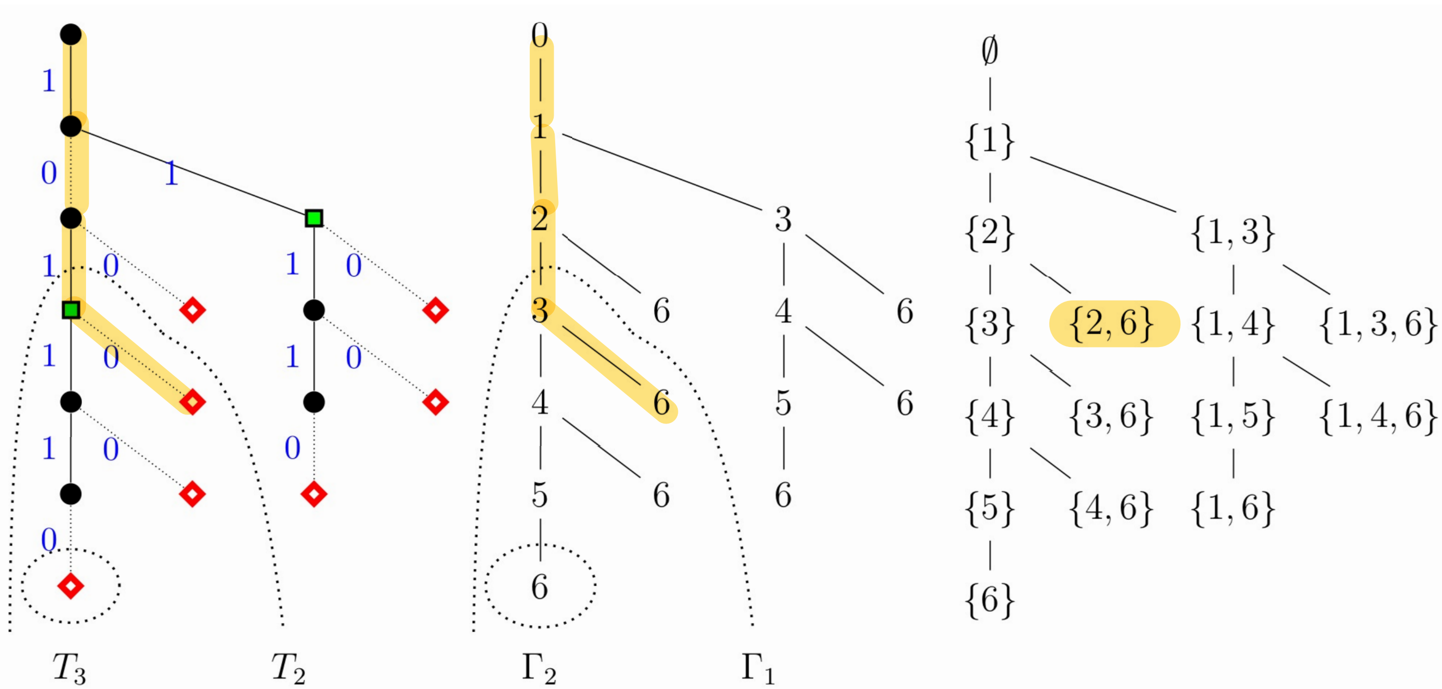
Perfect matchings of Snake Graph(Q(w)) ⟷ Subwords of w ⟷ antichains of Q(w)

down-sets of Q(w)

Ex: w = 1 0 1 1 1 0

Q(w) =




From "Combinatorics on words" paper [Leroy–Rigo–Stipulanti '17]

→ trie of subwords, antichain trie, antichains

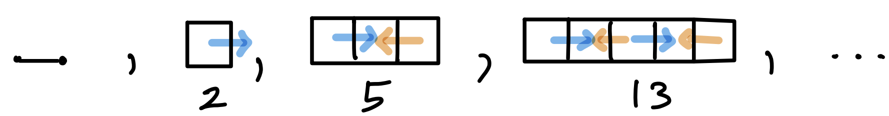
## Question: subwords of cyclic binary numbers & cyclic snake graphs

- Fibonacci numbers $F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$

  $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$

- Every-other Fibonacci numbers show up in computation of cluster variables of type $\tilde{A}_{1,1}$ $\quad 1 \rightrightarrows 2$

  Their snake graphs are of the form



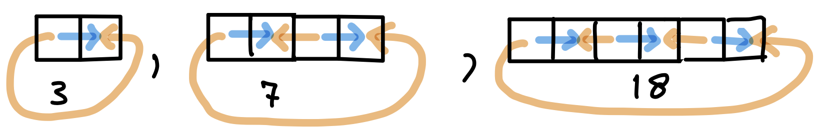

- In Benjamin-Quinn's model, Fibonacci numbers count tilings where the squares cannot be stacked

**Question: subwords of cyclic binary numbers & cyclic snake graphs**

- <u>Lucas numbers</u> $L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}$

  2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...

- Every-other Lucas numbers show up in computation of "bangles" and "bracelets" bases of type $\tilde{A}_{1,1}$ $1 \rightrightarrows 2$

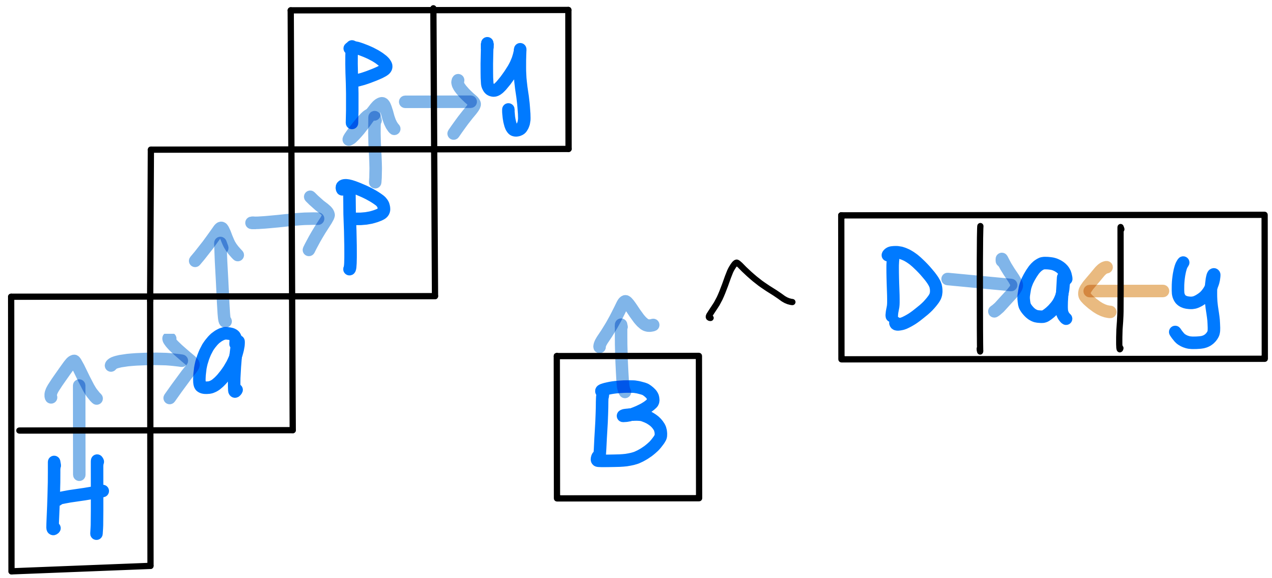
- They correspond to cyclic snake graphs (called band graph) of the form 

, ...

- In Benjamin-Quinn's model, the Lucas numbers count tilings of a circular board

In general, # pm's of band graph = # down-sets of a type $\tilde{A}_{p,q}$ quiver.

Find an analog of this in the settings of "circular" binary numbers




Jim!

$$64 = 2 ^ 6$$