Pattern-avoiding polytopes & Cambrian lattices

Jt w/ Esther Banaian, Sunita Chepuri, & Jianping Pan

12-pg summary: egunawan.github.io/polytope23.pdf

Tue, Jan 16 2024 Oberwolfach meeting "cluster algebras & its applications"

$$\begin{array}{ccccccc} n= 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

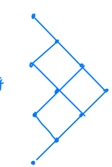
OEIS A003121 $(1, 1, 1, 2, 12, 286, 33592, \dots)$ counts:

a. longest chains (size $\binom{n+1}{2}$) in Tamari lattice (FN '12) (Fishel – Nelson)

b. (normalized) volume of "a pattern-avoiding subpolytope of the Birkhoff polytope" $B =$ Conv(perm. matrices avoiding 312 and 132) (DS '16) (Davis – Sagan)

c. linear extensions of a certain poset

ex $n=4$



(OEIS 3rd comment, '03)

D-S asked: Is B "equivalent" to the order polytope $\mathcal{O}(H)$ of a poset H?

"The ans Yes + more"

"Two poset polytopes" by Stanley '86:

- The order polytope $\mathcal{O}(P)$ of a poset P is the convex hull of the indicator vectors of order ideals of P

$W = A_n$ symmetric group gen'd by $s_1, \ldots, s_n$

<u>Main Thm</u> $\mathrm{Birk}(c)$ is equivalent to $\underbrace{\mathcal{O}(H)}$ where H is an AR quiver.

Volume = # linear extensions of H, $\dim = \binom{n+1}{2}$

<u>Cor</u>: volume of $\mathrm{Birk}(c)$ = # linear extensions of H

= # longest chains in c-Cambrian lattice

Ex: $W = A_4$

$c = s_1 s_4 s_2 s_3 = [1423]$, a Coxeter element

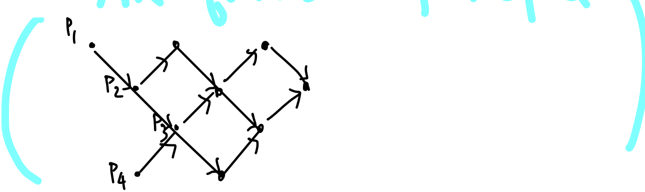
$$Q = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4$$

"the longest c-sorting word"

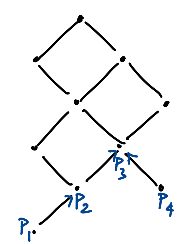
H = heap of $\underbrace{\mathrm{sort}_c(w_0)}$ = AR quiver of rep Q

[Cor of Bédard '99]

AR quiver of rep Q



$H =$



"Draw vertically"

Rotate 90°

TFAE:

1. w is <u>c-singleton</u> ← Def / Thm of [HLT '07] (Hohlweg - Lange - Thomas)

2. w is a prefix of a word in the <u>commutation class of</u> $sort_c(w_0)$

// (words obtained from $sort_c(w_0)$ by a sequence of swapping $ij \leftrightarrow ji$ if $|i-j| > 1$)

3. w avoids 4 patterns $31\underline{2}$, $\bar{2}31$, $13\underline{2}$, $\bar{2}13$ [Reading '04]

"For Tamari case, the 4 patterns collapse to just 2 patterns 312, 132

SKIP 4

4. w is a linear extension of an order ideal of H (cor of heap theory [Stembridge '96, Stanley EC vol 1, etc])

Def: ("The pattern-avoiding polytope")

$\mathrm{Birk}(c) := \mathrm{Conv}(\text{permutation matrices of } c\text{-singletons})$

$\dim = \binom{n+1}{2} = \left(\frac{(n+1)n}{2}\right)$, lives in dimension $(n+1)^2$