Pattern-avoiding polytopes and Cambrian lattices

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Our project was inspired by the OEIS sequence A003121 which counts shifted standard tableaux of staircase shape, longest chains in the Tamari lattice, linear extensions of a certain poset, and reduced words in a certain commutation class of the longest permutation w_0. Recently, it was shown by Davis and Sagan [1] that this sequence gives the normalized volume of a certain "pattern-avoiding polytope," a subpolytope of the Birkhoff polytope whose vertices are 132 and 312 avoiding permutations. Since these permutations form a distributive sublattice of the right weak order, Davis and Sagan asked whether their polytope might be unimodularly equivalent to the (Stanley's) order polytope ([4]) of a poset.

In our work, we associate a pattern-avoiding polytope $\mathrm{Birk}(c)$ to each Coxeter element c in the symmetric group and prove that $\mathrm{Birk}(c)$ is unimodularly equivalent to the order polytope of a poset. For the Coxeter element corresponding to the Tamari lattice, our result answers Davis and Sagan's question in the affirmative.

Example. Consider the type A_4 symmetric group W, and a Coxeter element $c = s_1 s_4 s_2 s_3$. This gives a quiver Q which is an orientation of the type A_4 Dynkin diagram. Consider the Auslander–Reiten quiver of $\mathrm{rep}\, Q$ drawn vertically, and let H be the "heap" obtained from it by replacing all representations in the τ^{-1}-orbit of $P(j)$ with the label s_j. See Figure 1.

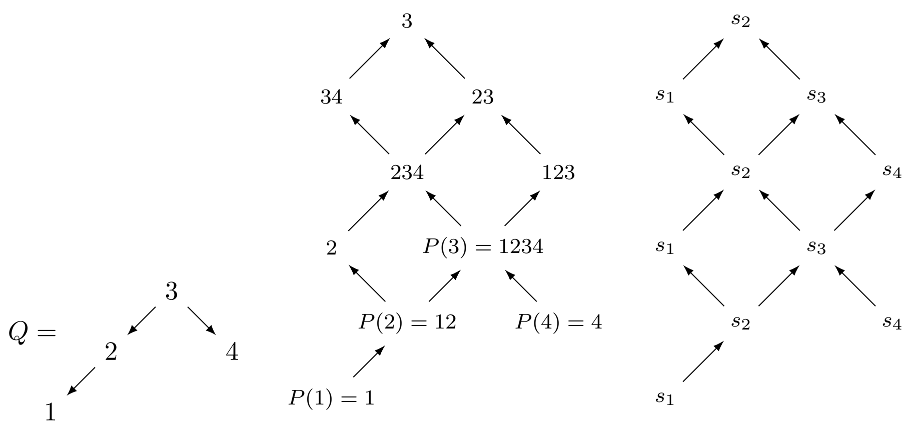

Figure 1. From left to right: Quiver Q; The Auslander–quiver of $\mathrm{rep}\, Q$; The heap H of $\mathrm{sort}_c(w_0)$

In general, H is the heap of the c-sorting word of w_0, denoted by $\mathrm{sort}_c(w_0)$. The c-singletons [2] are prefixes of words in the commutation class of $\mathrm{sort}_c(w_0)$; equivalently, order ideals of H; and permutations which avoid certain four patterns. For the "Tamari" c, these patterns collapse to just two patterns 132 and 312.

Definition. We define our "pattern-avoiding" polytope $\mathrm{Birk}(c)$ to be the convex hull of the permutation matrices of c-singletons.

Theorem. $\mathrm{Birk}(c)$ is unimodularly equivalent to the order polytope of H.

Corollary. The normalized volume of $\mathrm{Birk}(c)$ is equal to the number of linear extensions of H which is the number of longest chains in (Reading's [3]) c-Cambrian lattice. For our example c, this number is 41.

References

[1] Robert Davis and Bruce Sagan. Pattern-avoiding polytopes. *European J. Combin.*, 74:48–84, 2018.

[2] Christophe Hohlweg, Carsten E. M. C. Lange, and Hugh Thomas. Permutahedra and generalized associahedra. *Adv. Math.*, 226(1):608–640, 2011.

[3] Nathan Reading. Cambrian lattices. *Adv. Math.*, 205(2):313–353, 2006.

[4] Richard P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.