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Title: Pattern-avoiding polytopes & Cambrian lattices via the type A Auslander – Reiten quivers

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12-pg summary: egunawan.github.io/polytope23.pdf

I. Set up

$W = A_n$ symmetric group gen'd by $s_1, \dots, s_n$. $s_i = (i, i+1)$

Let c be a Coxeter elt of W, product of all n simple reflections, in any order.

Let w_0 denote the longest permutation, $\ell(w_0) = \binom{n+1}{2}$

Ex: $W = A_4$ gen'd by s_1, s_2, s_3, s_4

cycle notation

$c = s_1 s_4 s_2 s_3 = (\underline{2}\ \underline{3}\ 5\ \overline{4}\ 1)$

1-1 $\updownarrow$

$Q = \text{Quiver}(c) = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4$

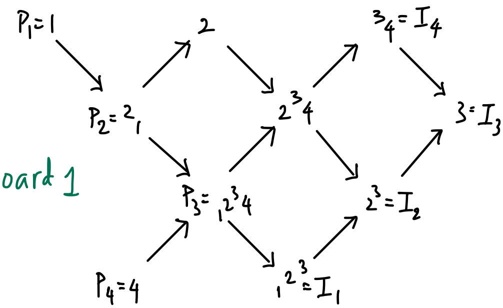
Rem: choosing a Coxeter elt is equivalent to choosing an orientation of the corresponding Dynkin diagram

Rule: $k \leftarrow k+1$ if k appears left of $k+1$ in c

$k \rightarrow k+1$ otherwise

$\ell(w_0) = \binom{5}{2} = 10$

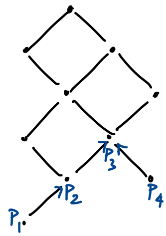
Γ = AR quiver of rep Q



Board 1

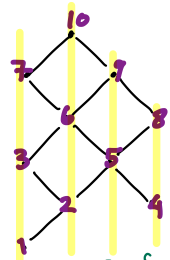
"Draw vertically"

$\Gamma =$



Board 2

Label w/ simple reflections



Let $H :=$ poset on $\{1, \dots, \ell(w_0)\}$ whose Hasse diagram is Γ with labels $s_1, \dots, s_n$

<u>Main Result</u> We define a "pattern avoiding" polytope $\mathrm{Birk}(c)$ and prove that it is unimodularly equivalent to the (Stanley's) order polytope $\mathcal{O}(H)$, where H is the Auslander–Reiten quiver of rep Q.

<u>Cor</u>: Volume of $\mathrm{Birk}(c)$ = # linear extensions of H

= # longest chains in the <u>c-Cambrian lattice</u>

- its Hasse diagram is an oriented exchange graph of the cluster algebra with initial quiver $Q = \mathrm{Quiver}(c)$
- a <u>Tamari lattice</u> if $c = s_1 s_2 \dots s_n$

(a partial order on ways to use parentheses)

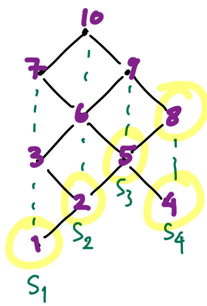
II. The order polytope

The <u>order polytope</u> $\mathcal{O}(H)$ of a finite poset H is

$$\mathrm{Conv}\{\text{indicator vectors of } I \mid I \text{ is an order ideal of } H\}$$

"the convex hull of the indicator vectors of order ideals of P"

Ex:

$I =$ 

indicator vector of I = $\begin{array}{c} 1 \\ 2 \\ \\ 4 \\ 5 \\ \\ \\ 8 \\ \\ \\ \end{array}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathcal{O}(H) = \left\{ \vec{x} \in \mathbb{R}^{10} : \begin{array}{l} 0 \le x(i) \le 1 \text{ for all } i = 1, \dots, 10 \text{ and} \\ x(i) \le x(j) \text{ whenever } \underbrace{i \lessdot j}_{\text{poset relation}} \end{array} \right\}$$

- $\dim \mathcal{O}(H) = \# H$
- Vertices of $\mathcal{O}(H) \longleftrightarrow$ order ideals
- $\underbrace{\text{Normalized volume}}_{\dim(\text{polytope})!\ \mathrm{Vol}(\text{polytope})}$: # <u>linear extensions</u> of H

A total order $\pi = \pi(1) \dots \pi(\ell(w_0))$ that is consistent w/ the structure of H, i.e. $x \lessdot_H y$ implies $\pi(x) < \pi(y)$

Ex: $\mathrm{Dim}\ \mathcal{O}(H) = 10$

Volume = 41

$\boxed{\text{III. c-Singletons}}$

TFAE:

1. w is <u>c-singleton</u> $\leftarrow$ Def / Thm of (Hohlweg – Lange – Thomas '07)

pronounced like "Holweg" "Lang-e"

$+$

heap theory

2. w corresponds to an order ideal I of H,

i.e w has a reduced word which is a linear extension of I.

Ex: $I = \{1, 2, 4, 5, 8\}$

$w(I) = s_1 \, s_2 \, s_4 \, s_3 \, s_4$

3. w avoids four certain patterns (for all n) $\quad 31\underline{2}$, $\overline{2}31$, $13\underline{2}$, $\overline{2}13$ (Reading '04)

Rem:

If $c = s_1 s_2 \ldots s_n$ then w is c-singleton iff w avoids 312, 132

$Q = 1 \leftarrow 2 \leftarrow \cdots$

(the "Tamari case")

"the four patterns collapse to just two patterns"

IV. A "pattern-avoiding" Birkhoff subpolytope

Def (Our pattern-avoiding polytope):

- $\mathrm{Birk}(c) := \mathrm{Conv}\left(\begin{array}{c}\text{permutation matrices } M(w) \\ \text{of } c\text{-singletons } w\end{array}\right)$

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \;\middle|\; \sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i \geq 0, \; x_i \in X \right\}$$

- The permutation matrix $M(w)$ of w is the $(n\times 1)\times(n+1)$ matrix s.t. row i, col j has entry $\begin{cases} 1 & \text{if } w(i)=j \\ 0 & \text{otherwise} \end{cases}$

Ex: $I=\{1,2,3\}$, $w(I) = s_1 s_2 s_1 = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{array}$ 2-line notation

$$M(32145) = \begin{array}{|c|c|c|c|c|} \hline & & \boxed{3} & & \\ \hline & 2 & & & \\ \hline 1 & & & & \\ \hline & & & 4 & \\ \hline & & & & 5 \\ \hline \end{array} \rightarrow \text{vector in } \mathbb{R}^{25}$$

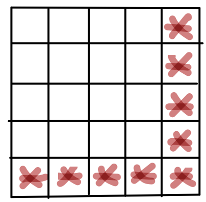
Rem Birk(C) lives in dimension $(n+1)^2 = 5^2 = 25$

Prop Birk(C) has $\dim = \binom{n+1}{2} = \frac{(n+1)n}{2}, = \frac{(5)4}{2} = 10$

Fifteen Relations that give us dim 10:

(Birkhoff relation)

- Each row and col sum up to 1:

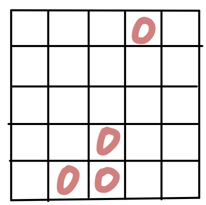


minus 9

From pattern avoidance conditions:

(Zero relation)

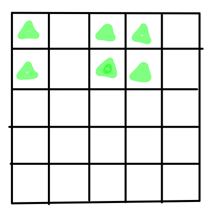
- The pattern avoidance conditions immediately require these four entries to be 0:



minus 4

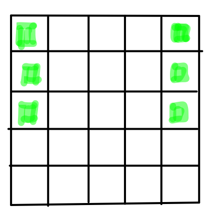
- (Summing relation)

(i) These entries must sum up to 1:



minus 1

(ii) These entries must sum up to 1:



minus 1

+

minus 15

$\boxed{\text{V. } \mathrm{Birk}(c) \cong \mathcal{O}(H)}$

unimodularly equivalent

Projection π_c : $(n+1)\times(n+1)$ 5x5 matrices in $\mathrm{Birk}(c) \longrightarrow \mathbb{R}^{10}$ $\mathbb{R}^{\binom{n+1}{2}}$

$c = (\bar{4}\ 1\ \underline{2}\ \underline{3}\ 5)$ $\qquad Q = \mathrm{Quiver}(c) = 1 \leftarrow \underline{2} \leftarrow \underline{3} \rightarrow \bar{4}$

Do: col $\underline{2}$: $(1,2)^{(10)}$

col $\underline{3}$: $(2,3)^{(9)}, (1,3)^{(8)}$

col $n+1$: $(4,5)^{(7)}, (3,5)^{(6)}, (2,5)^{(5)}, (1,5)^{(4)}$

$$\begin{array}{cccccc} & & \underline{2} & \underline{3} & & \\ & \cdot & 10 & 8 & 0 & 4 \\ & & \cdot & 9 & 1 & 5 \\ & & & \cdot & 2 & 6 \\ & & & 0 & \cdot & 7 \\ & 3 & 0 & 0 & & \cdot \\ \text{also} & & & & \bar{4} & \end{array}$$

Then do: $u = \bar{4}$, $\quad m = \min(u-1, n+1-u) = \min(3, 5-4) = 1$

• First: $\underbrace{(n+1, c^1(u))}_{(5,\ c(4)) = (5,1)^{(3)}}, (n, c^2(u)), \ldots, (n+2-m, c^m(u))$

• Then, if $u-1 > m$, do: $\overset{4}{u} - 1 - \overset{1}{m} = 2$ entries:

$\underbrace{(u-1, u), (u-2, u)}_{(3,4)^{(2)}, (2,4)^{(1)}}, \ldots, (m+1, u)$

<u>Thm</u> π_c sends integral points to integral points

"$\mathrm{Birk}(c)$ has no internal integral points"

<u>Thm</u> Fix Coxeter elt c.

There exists a unique $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix $\mathcal{U}_c$ with 1's on the main diagonal (in particular, $\mathcal{U}_c$ has det 1)

s.t $\mathcal{U}_c \circ \pi_c(M(w))$

= indicator vector of the order ideal I of H corresponding to w

for all c-singletons w. □

VI. Inspiration

$n=$ 0 1 2 3 4 5 6

OEIS A003121 $(1, 1, 1, 2, 12, 286, 33592, \ldots)$ counts:

a. longest chains (length $\binom{n+1}{2}$) in Tamari lattice

(Fishel-Nelson '12)

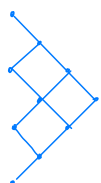
b. (normalized) volume of $B = \mathrm{Conv}($perm. matrices avoiding 312 and 132$)$

(Davis-Sagan '16)

c. linear extensions of poset H whose Hasse diagram is the Auslander-Reiten quiver of $\mathrm{rep}(\leftarrow \leftarrow \cdots \leftarrow)$

(OEIS 3rd comment '03)

ex $n=4$



d. (normalized) volume of the order polytope $\mathcal{O}(H)$

VII. Questions

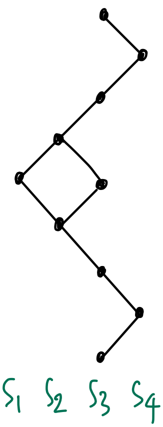
① Our H is the heap of the c-sorting word of w_0.
If we take an arbitrary reduced word $[u]$ of an arbitrary permutation w, and let $H = \mathrm{Heap}([u])$, when is $\mathcal{O}(H) \cong \mathrm{Conv}($perm. matrices of order ideals of $H)$?

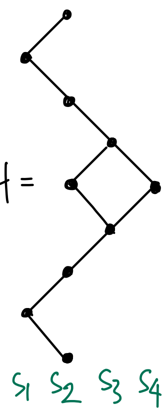
Ans Not in general.

Two counterexamples in A_4:

More counterex in A_5 and bigger A_n

$[u]=[3\ 4\ 3\ 2\ 3\ 1\ 2\ 3\ 4\ 3]$ $[u]=[2\ 1\ 2\ 3\ 2\ 4\ 3\ 2\ 1\ 2]$

$H=$ 

$H=$ 

$\dim \mathcal{O}(H) = 10$, $\dim \mathrm{Birk}(H) = 9$, so $\mathcal{O}(H) \not\cong \mathrm{Birk}(H)$.

② Relate our results to rep theory meaningfully

③ Generalize to type BDEFGHI ?

$\#$ linear extensions

A_4 41 our ex

12 $\cdot \to \cdot \to \to$ OEIS A003121

70 $\to \leftarrow \to$