Triangulations & maximal almost rigid modules over gentle algebras

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FD Seminar

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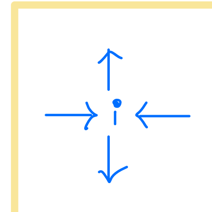
Gentle algebras

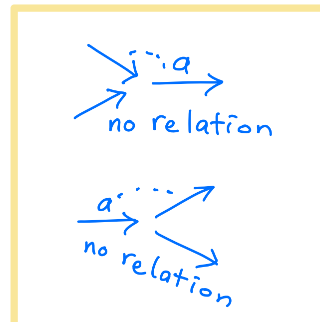
Def A finite-dimensional algebra $A = \mathbb{k}Q/I$ is *gentle* if :

(G1) $\forall$ vertex i of Q, $\exists$ at most 2 arrows starting at i
$\exists$ at most 2 arrows ending at i



(G2) I is generated by paths of length 2

(G3) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b s.t $ba \notin I$
$\exists$ at most 1 arrow c s.t $ac \notin I$



(G4) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b' s.t $b'a \in I$
$\exists$ at most 1 arrow c' s.t $ac' \in I$

Ex

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$\boxed{\mathbb{k}Q \text{ Not gentle}}$

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

$\boxed{\mathbb{k}Q/I \text{ is gentle}}$

$$Q = \begin{tikzcd} & 4 \arrow[dl, "c"'] & \\ 3 & 2 \arrow[l, "d"'] \arrow[r, "a"'] & 1 \arrow[ul, "b"'] \end{tikzcd}$$

$I = \langle ab, ca \rangle$

$\boxed{\mathbb{k}Q/I \text{ is also gentle}}$

String modules

A <u>string</u> ω is a walk along the arrows in $Q_1 \cup \underbrace{Q_1^{-1}}_{\{\text{opposite arrow } \alpha^{-1} \mid \alpha \in Q_1\}}$

with
- no backtrack $a a^{-1}$ or $a^{-1} a$
- no subwalk v with $v \in I$ or $v^{-1} \in I$
  (no going through relations)

<u>Ex</u>

$$Q = \begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

- ac is <u>a string</u>, $b^{-1}c$ is <u>a string</u>, ab is <u>not a string</u>

$$\begin{tikzcd}[row sep=small] 1 \arrow[r, "a"] & 2 \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & 3 \\ 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & 4 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] & \end{tikzcd}$$

- e_1, e_2, e_3, e_4 are <u>trivial strings</u>

[Butler – Ringel 1987]

If $\Bbbk Q/I$ is a gentle algebra,

indecomposable modules are "string modules" and "band modules"

ω string $\longleftrightarrow$ $M(\omega)$ string module

Recall def $T \in \mathrm{mod}(A)$ is *tilting* if

(T1) For each pair A, B of summands of T,

if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$

(T2) T is maximal with respect to (T1)

(T3) T has projective dimension at most 1

Rem A tilting T has $\underbrace{|Q_0|}_{\#\text{ of vertices of } Q}$ non-isomorphic summands

Def $A = \Bbbk Q/I$ gentle algebra. A basic direct sum $T \in \mathrm{mod}(A)$ of string modules is *maximal almost rigid* (mar) if

(M1) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence,
then $E \cong B \oplus A$ or E is indecomposable

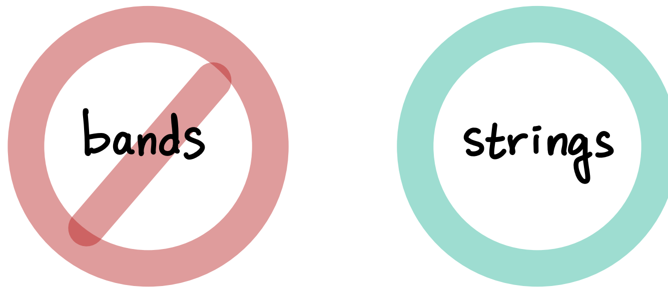
called *almost rigid*

(M2) T is maximal with respect to (M1)

Originally defined for type A $\Bbbk Q$ by [Barnard, G., Meehan, Schiffler]

Rem (M2) can be replaced with:
"T has $\underbrace{|Q_0|}_{\#\text{ of vertices of } Q} + \underbrace{|Q_1|}_{\#\text{ of arrows of } Q}$ non-isomorphic summands"

<u>Rem</u> Question: Why do we restrict to only string modules?



Answer: there are band modules that satisfy (M1)

Ex: $\mathbb{k} \underset{\lambda}{\overset{1}{\rightrightarrows}} \mathbb{k}$ satisfies (M1).

- If $\lambda \neq \lambda'$, then $\mathrm{Ext}^1\left(\mathbb{k} \underset{\lambda}{\overset{1}{\rightrightarrows}} \mathbb{k}, \mathbb{k} \underset{\lambda'}{\overset{1}{\rightrightarrows}} \mathbb{k}\right) = 0$.
- If we allow bands, we would have infinitely many summands.

[Opper – Plamondon – Schroll 2018] + [Baur – Coelho Simões 2018]

extra marked points *

Tiling or dissection (S, M, P, M^*)

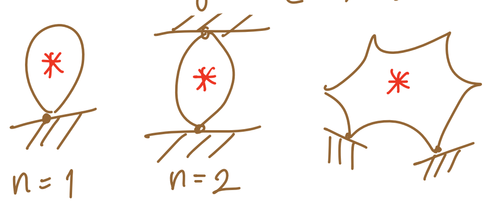
S: Surface w/ nonempty ∂S

M: Marked points in ∂S

P: Dissection of (S, M) into tiles

- n-gon $(n \geqslant 2)$ w/ 1 bdry edge as its side

- internal n-gon $(n \geqslant 1)$

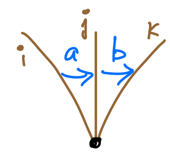


M^*: Put a point * in each tile

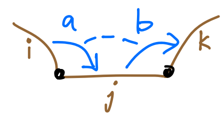
- on bdry edge between points of M
- in interior of internal disk

$\bullet \in M \qquad * \in M^*$

Rules :

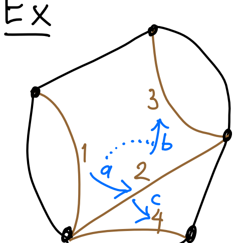


of P $\longleftrightarrow$ $i \xrightarrow{a} j \xrightarrow{b} k \quad (ab \notin I)$



of P $\longleftrightarrow$ $i \xrightarrow{a} j \xrightarrow{b} k \quad (ab \in I)$

Ex



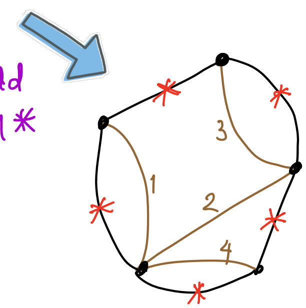
(S, M, P)

$\longleftrightarrow$

$$Q = \begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[dr, "c"'] & 3 \\ & & 4 \end{tikzcd}$$

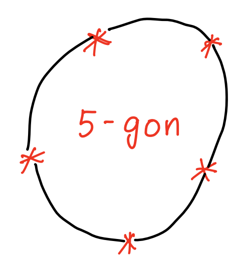
$$I = \langle ab \rangle$$

Add M^*



(S, M, P, M^*)

Erase M, P



(S, M^*)

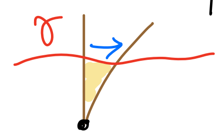
[Opper – Plamondon – Schroll 2018] + [Baur – Coelho Simões 2018]

extra marked points $*$

string modules of $\Bbbk Q/I$ $\overset{1-1}{\longleftrightarrow}$ permissible arcs γ in S:

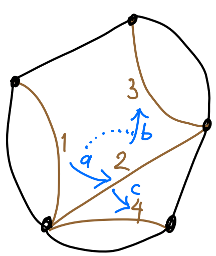
(i) endpoints are in M^*

(ii) each pair of consecutive crossings of γ and P corresponds to an arrow of Q



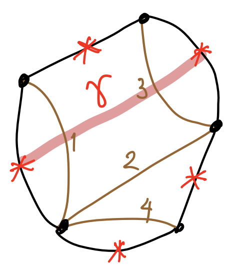
(locally cuts up a triangle)

Ex (S, M, P)

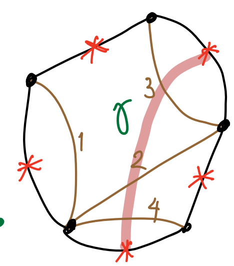


$$Q = \begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[dr, "c"'] & 3 \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

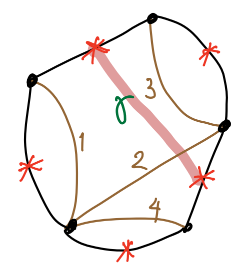


- γ is not permissible
- Consecutive crossings arc 1, arc 3 do not correspond to an arrow of Q



permissible γ

$\leftrightarrow$ string $b^{-1}c$ or $c^{-1}b$



permissible γ

$\leftrightarrow$ trivial string e_2

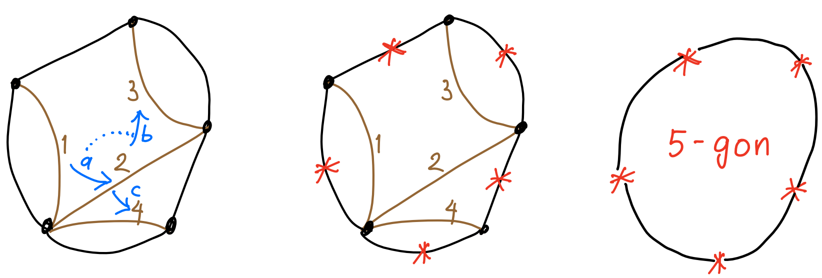
Thm 1 (Barnard – Coelho Simões – G. – Schiffler [B.CS.G.S])

$$\left\{ \begin{array}{c}\text{mar modules} \\ \text{mar}(A)\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l}\text{permissible ideal triangulations of } (S, M^*) \\ \text{including boundary edges}\end{array} \right\}$$

Ex

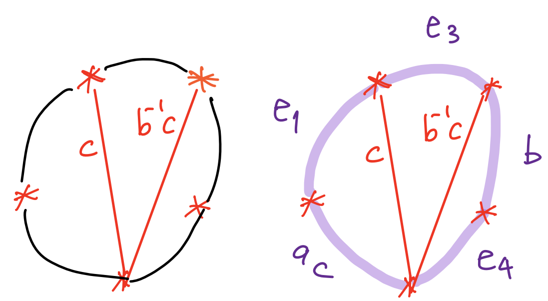
$$Q = \begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$



$(S, M, P) \Rightarrow (S, M, P, M^*) \Rightarrow (S, M^*)$

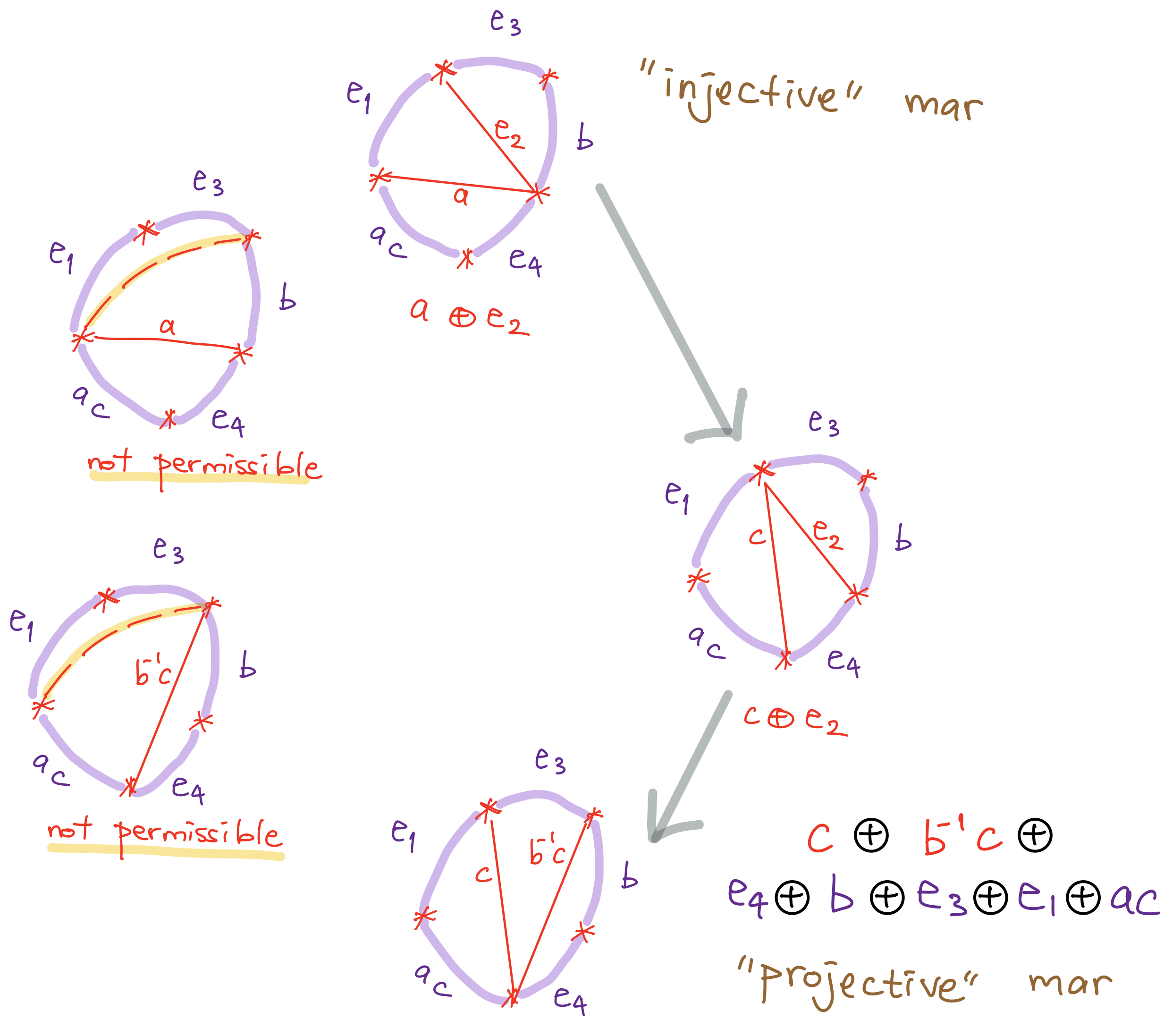
Recall: Each T in $\text{mar}(A)$ has $|Q_0| + |Q_1|$ summands
$\quad 4 + 3$



$$c \oplus b^{-1}c \oplus \underbrace{e_4 \oplus b \oplus e_3 \oplus e_1 \oplus ac}_{\substack{\text{These 5 summands} \\ \text{are required in} \\ \text{every } T \in \text{mar}(A)}} \in \text{mar}(A)$$

Ex There are exactly three mar modules for

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad I = \langle ab \rangle$$



Thm 2

[B.CS.G.S]

Construct a new gentle algebra

$\bar{A} := \Bbbk\bar{Q}/\bar{I}$ where $|\bar{Q}_0| = |Q_0| + |Q_1|$.

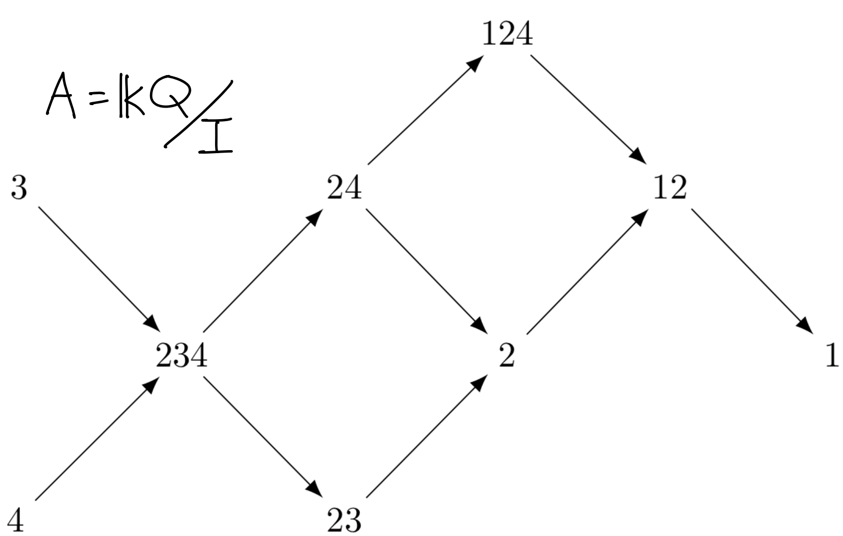
Then $T \in \operatorname{mar}(A) \Rightarrow \operatorname{End}_A(T) \cong \operatorname{End}_{\bar{A}}(\bar{T})$

where $\bar{T}$ is tilting in $\operatorname{mod}(\bar{A})$.

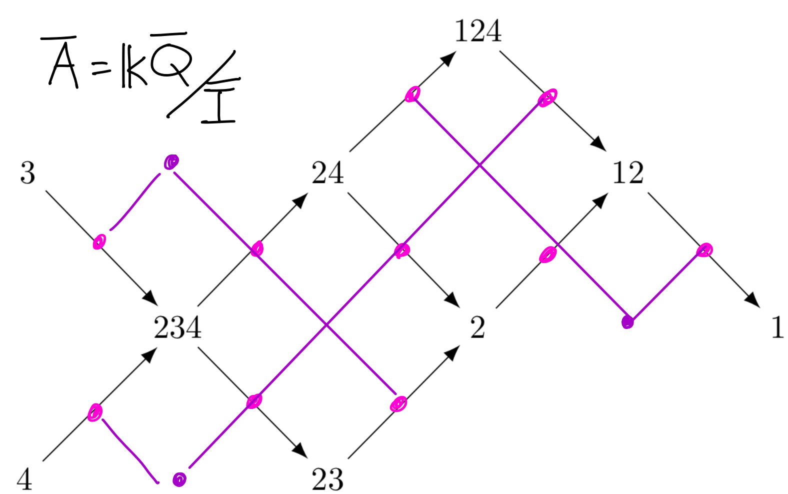
$$Q = 1 \xrightarrow{a} 2 \quad 2 \xrightarrow{b} 3 \quad 2 \xrightarrow{c} 4 \qquad I = \langle ab \rangle$$

$$\bar{Q} = 1 \xrightarrow{a_1} a \xrightarrow{a_2} 2 \xrightarrow{b_1} b \rightarrow 3 \quad 2 \rightarrow c \rightarrow 4 \qquad \bar{I} = \langle a_2 b_1 \rangle$$

$A = \Bbbk Q/I$



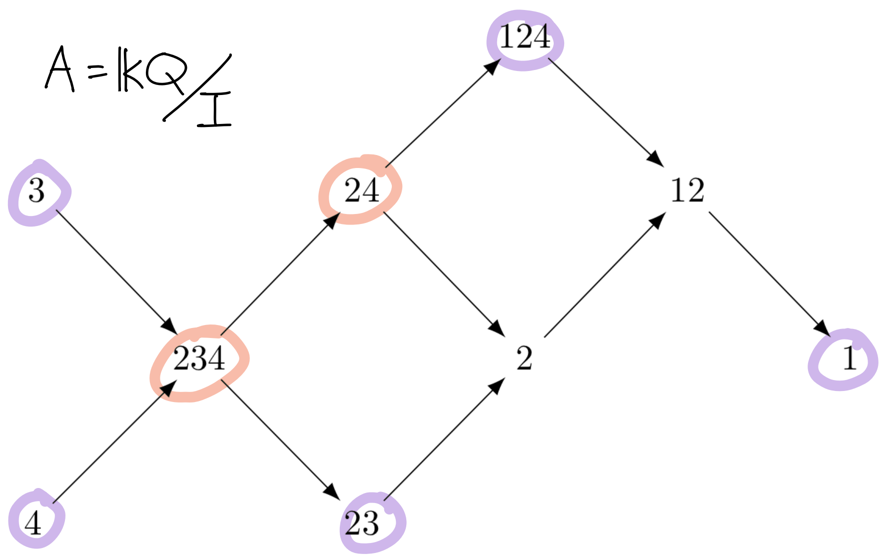
$\bar{A} = \Bbbk\bar{Q}/\bar{I}$



$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3,\quad 2 \xrightarrow{c} 4$ $I = \langle ab \rangle$

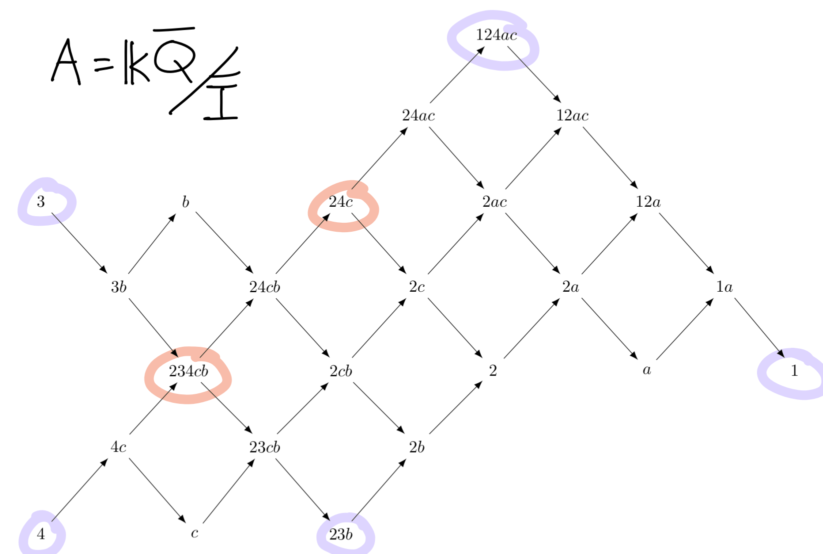
$\overline{Q} = 1 \xrightarrow{a_1} a \xrightarrow{a_2} 2 \xrightarrow{b_1} b \rightarrow 3,\quad 2 \rightarrow c \rightarrow 4$ $\overline{I} = \langle a_2 b_1 \rangle$

$A = \Bbbk Q / I$



$T \in \mathrm{mar}(A)$

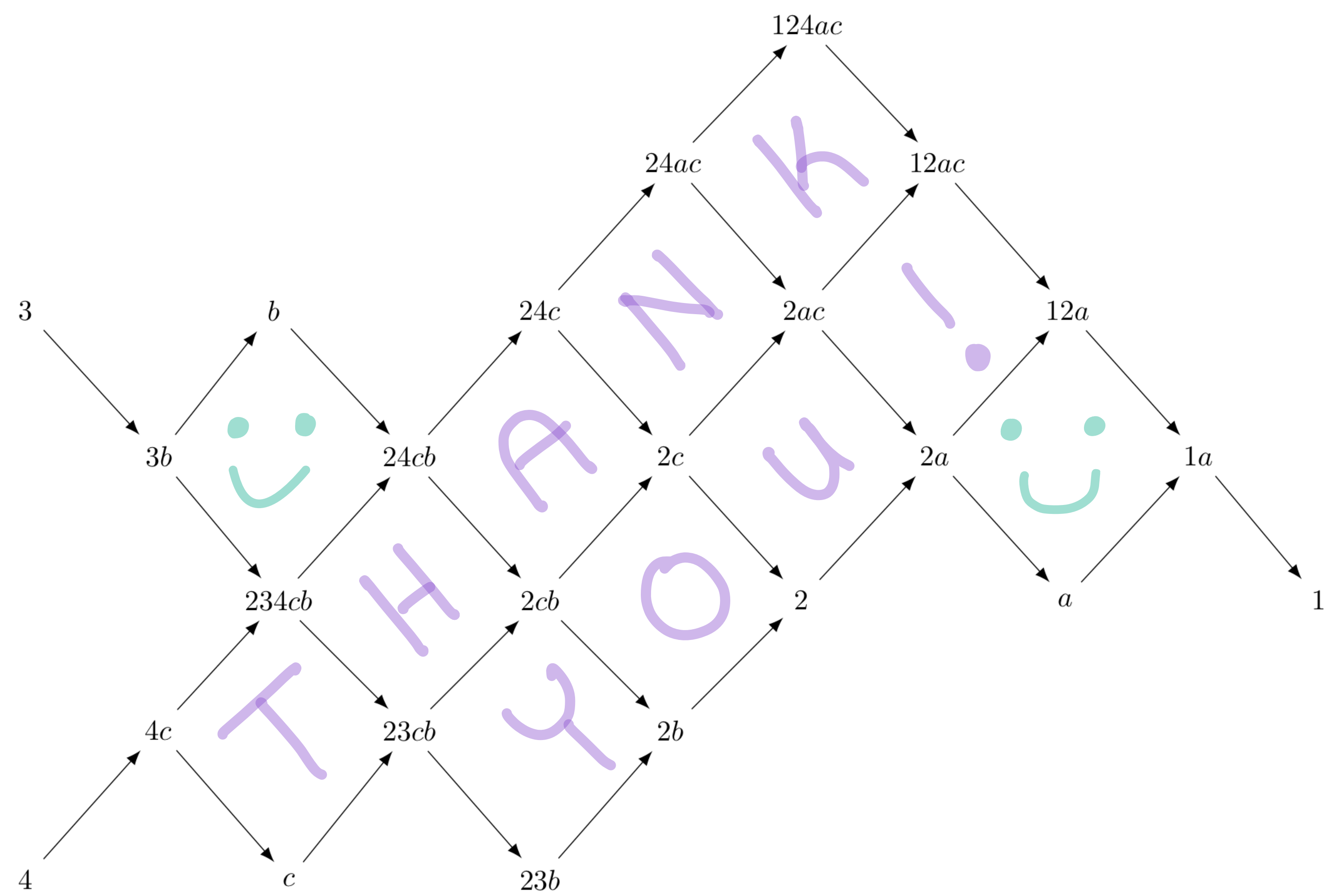
$A = \Bbbk \overline{Q} / I$



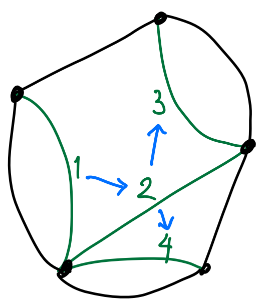
$\overline{T}$ is a tilting module over $\overline{A}$

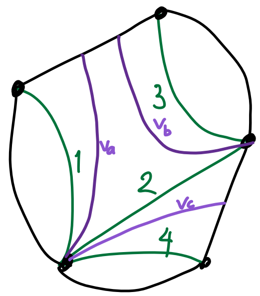
mar over A tilting over $\overline{A}$

<u>Thm 2</u> $\mathrm{End}_A(T) \cong \mathrm{End}_{\overline{A}}(\overline{T})$


124ac
24ac
12ac
3
b
24c
2ac
12a
3b
24cb
2c
2a
1a
234cb
2cb
2
a
1
4c
23cb
2b
4
c
23b
THANK YOU!

Dissection of $\bar{A}$ (Extra)





(S, M, P)

$\updownarrow$

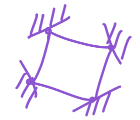
A

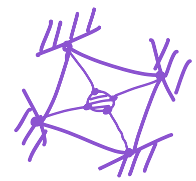
Add an extra arc in P for each arrow in Q_1

$(\bar{S}, \bar{M}, \bar{P})$

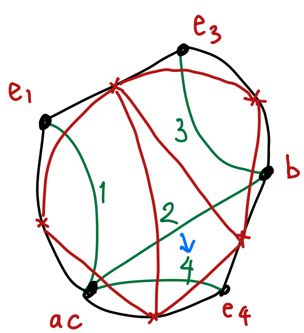
$\updownarrow$

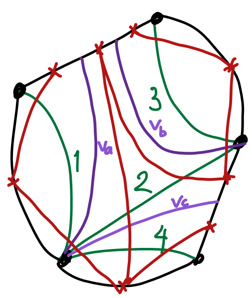
$\bar{A}$

If P has an internal n-gon tile  ,

replace with annulus 

$$\operatorname{mod} A \xrightarrow{G} \operatorname{mod} \bar{A}$$

$\gamma =$ 



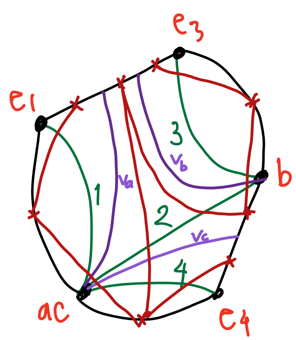
$= G(\gamma)$

$$T = c \oplus e_2 \oplus \underbrace{e_1 \oplus e_3 \oplus b \oplus e_4 \oplus ac}_{\text{Required summands}} \in \operatorname{mar} A$$

Dissection of $End_A(T)$ (Extra)

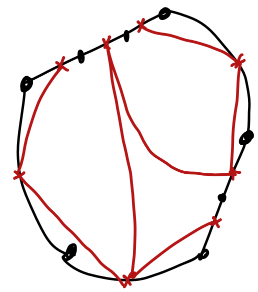
Q: What is the partial triangulation $G(\Upsilon)$?

Ans/Thm 3: We can construct tiling for $End_A(T)$ as follows



$= G(\Upsilon)$

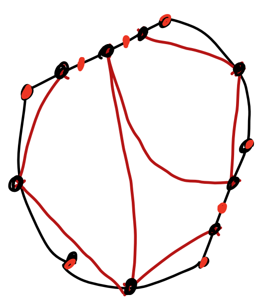
Erase $\overline{P}$ $\longrightarrow$

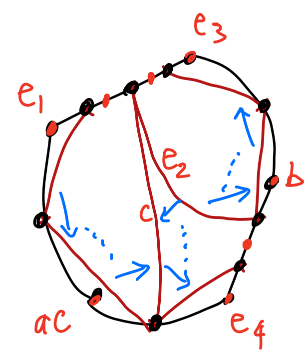


$= G(\Upsilon)$

swap M & M* in ∂S $\longrightarrow$

Note: No puncture by construction of tiling of $\overline{A}$





$$\begin{tikzcd}
e_1 \arrow[dr] & & e_2 \arrow[r] \arrow[d] & b \arrow[r] & e_3 \\
& ac \arrow[r] & c \arrow[r] & e_4
\end{tikzcd}$$

is the endomorphism algebra

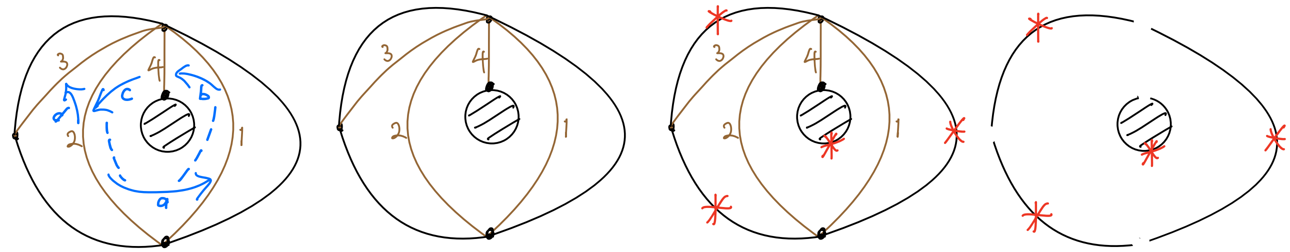
$End_A(T) \cong End_{\overline{A}}(\overline{T})$

Example (extra)

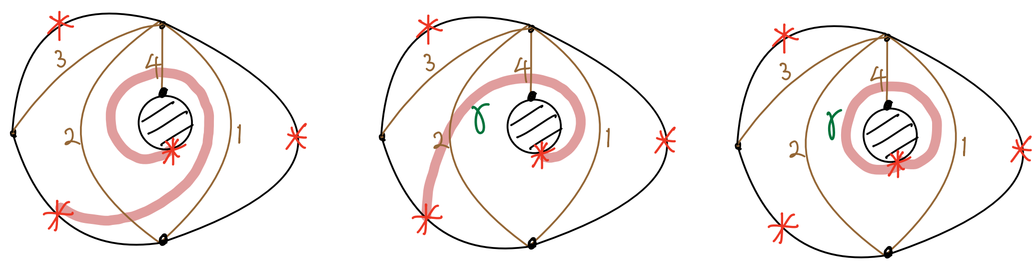
$$Q = \begin{tikzcd} & & 4 \arrow[dl, "c"'] & \\ 3 & 2 \arrow[l, "d"'] \arrow[rr, "a"'] & & 1 \arrow[ul, "b"'] \end{tikzcd}$$

$I = \langle ab, ca \rangle$

(S, M, P) (S, M, P) (S, M, P, M^*) (S, M^*)



An annulus with
3 points on one bdry
& 1 point on the other



Not permissible
because
the crossings
w/ P at
arc 2, arc 4
do not
correspond
to an arrow in Q

permissible
$\gamma \longleftrightarrow$
string $4 \xrightarrow{c} 2$

permissible
$\gamma \longleftrightarrow$
trivial
string e_4