Box-ball systems, RSK tableaux, and the Motzkin numbers

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Solitary waves (solitons)

Scott Russell's first encounter (August 1834)

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped.

[The mass of water in the channel] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, ... and after a chase of one or two miles I lost it in the windings of the channel."

Soliton on the Scott Russell Aqueduct on the Union Canal (July 1995)

(ma.hw.ac.uk/solitons/press.html)

Two soliton animation: www.desmos.com/calculator/86loplpajr

Permutations

Let S_n denote the set of permutations on the numbers $\{1, \ldots, n\}$.

We will represent permutations in *one-line notation*, as

$$w = w(1)\, w(2)\, \cdots\, w(n) \in S_n.$$

Example

A permutation in S_6 in one-line notation: 452361

(Multicolor) box-ball system, Takahashi 1993

A *box-ball system* is a dynamical system of box-ball configurations.

- At each configuration, balls are labeled by numbers 1 through n in an infinite strip of boxes.
- Each box can fit at most one ball.

Example

A possible box-ball configuration:

$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**									$\cdots$

Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.

$t = 0$	$\cdots$	4	5	2	3	6	1				$\cdots$
	$\cdots$	4	5	2	3	6		1			$\cdots$
	$\cdots$	4	5		3	6	2	1			$\cdots$
	$\cdots$	4	5			6	2	1	3		$\cdots$
	$\cdots$		5	4		6	2	1	3		$\cdots$
	$\cdots$			4	5	6	2	1	3		$\cdots$
$t = 1$	$\cdots$			4	5		2	1	3	6	$\cdots$

Box-ball moves ($t = 0$ through $t = 5$)

$t = 0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**															$\cdots$
$t = 1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**												$\cdots$
$t = 2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**									$\cdots$
$t = 3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**						$\cdots$
$t = 4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**			$\cdots$
$t = 5$	$\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future box-ball moves.

Example

The strings **4**, **25**, and **136** are solitons:

$t = 3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**							$\cdots$
$t = 4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**				$\cdots$
$t = 5$	$\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

After a finite number of box-ball moves, the system reaches a *steady state* where:

- each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

Question (steady-state time)

The time when a permutation w first reaches steady state is called the *steady-state time* of w.

- Find a formula to compute the steady-state time of a permutation, without needing to run box-ball moves.

Tableaux (English notation)

Definition

- A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

1	2	4
3	5	
6	7	

1	3	6
2	5	
4		

1	3	4
2	7	
5	8	
6		

Nonstandard Tableau:

1	2	3
5	6	7
4		

Soliton decomposition

Definition

To construct *soliton decomposition* SD(w) of w, start with the one-line notation of w, and run box-ball moves until we reach a steady state; the 1st row of SD(w) is the rightmost soliton, the 2nd row of SD(w) is the next rightmost soliton, and so on.

Example

$t=0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**													$\cdots$
$t=1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**										$\cdots$
$t=2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**							$\cdots$
$t=3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**				$\cdots$
$t=4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**		$\cdots$

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \text{ with shape } (3,2,1).$$

RSK bijection

The classical Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $w = \mathbf{452361}$. Then

$$\mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \quad \text{and} \quad \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} .$$

RSK bijection example

Let $w = \mathbf{452361}$.

$$\text{P}: \quad \mathbf{4} \qquad \begin{array}{cc} 4 & \mathbf{5} \end{array} \qquad \begin{array}{cc} \mathbf{2} & 5 \\ 4 & \end{array} \qquad \begin{array}{cc} 2 & \mathbf{3} \\ 4 & 5 \end{array} \qquad \begin{array}{ccc} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{array} \qquad \begin{array}{ccc} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \qquad \text{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$\text{Q}: \quad \mathbf{1} \qquad \begin{array}{cc} 1 & \mathbf{2} \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \mathbf{3} & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & \mathbf{4} \end{array} \qquad \begin{array}{ccc} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{array} \qquad \text{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Insertion and bumping rule for P

- Insert x into the first row of P.
- If x is larger than every element in the first row, add x to the end of the first row.
- If not, replace the smallest number larger than x in row 1 with x. Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

Q(w) determines the box-ball dynamics of w

Theorem (2021)

If $\mathrm{Q}(v) = \mathrm{Q}(w)$, then

- v and w first reach steady state at the same time, and
- the soliton decompositions of v and w have the same shape

Example

$$v = 21435 \text{ and } w = 31425$$

$$\mathrm{Q}(v) = \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array}$$

Both v and w have steady-state time $t = 1$

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & & \\ \cline{1-1} 3 & & \\ \cline{1-1} \end{array}$$

Questions (steady-state time)

Two permutations are said to be *Q-equivalent* if they have the same Q-tableau.

- Given a Q-tableau, find a formula to compute the steady-state time for all permutations in this Q-tableau equivalence class.
- Find an upper bound for steady-state times of all permutations in S_n.

L-shaped soliton decompositions

Theorem (2021)

If a permutation has an L-shaped soliton decomposition $\begin{array}{|c|c|c|l}\hline 1 & & & \cdots \\ \cline{1-3} & \multicolumn{3}{l}{} \\ \cline{1-1} & \multicolumn{3}{l}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{3}{l}{} \end{array}$,

then its steady-state time is either $t = 0$ or $t = 1$.

Remark

Such permutations include "noncrossing involutions" and "column words" of standard tableaux.

Example

Both $v = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$v = (12)(34)$ and $w = 31425$ is the column word of $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$.

Maximum steady-state time

Theorem (UConn 2020)

If $n \geq 5$ and

$$Q(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1} \end{array},$$

then the steady-state time of w is $n-3$.

Conjecture

For $n \geq 4$, the steady-state time of a permutation in S_n is at most $n-3$.

A permutation with steady-state time $n-3$

Let $w = 452361 \in S_6$. Then $\mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1}\end{array}$ and the steady-state time of w is $3 = n-3$.

$t=0$ $\cdots$	**4**	**5**	**2**	**3**	**6**	**1**																$\cdots$
$t=1$ $\cdots$			**4**	**5**		**2**	**1**	**3**	**6**													$\cdots$
$t=2$ $\cdots$					**4**	**5**	**2**			**1**	**3**	**6**										$\cdots$
$t=3$ $\cdots$							**4**	**2**	**5**				**1**	**3**	**6**							$\cdots$
$t=4$ $\cdots$								**4**		**2**	**5**					**1**	**3**	**6**				$\cdots$
$t=5$ $\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

Question (soliton decompositions)

- When is the soliton decomposition SD a standard tableau?

When is SD(w) a standard tableau?

Example

$$\mathrm{SD}(452361) = \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \quad \mathrm{SD}(21435) = \begin{array}{ccc} 1 & 3 & 5 \\ 4 & & \\ 2 & & \end{array} \quad \mathrm{SD}(31425) = \begin{array}{ccc} 1 & 2 & 5 \\ 4 & & \\ 3 & & \end{array}$$

Theorem (2020)

Given a permutation w, the following are equivalent:

1. $\mathrm{SD}(w)$ is standard
2. $\mathrm{SD}(w) = \mathrm{P}(w)$
3. the shape of $\mathrm{SD}(w)$ is equal to the shape of $\mathrm{P}(w)$

Definition (good permutations)

We say that a permutation w is *good* if the tableau $\mathrm{SD}(w)$ is standard.

$\mathrm{Q}(w)$ determines whether w is good

Proposition

Given a standard tableau T, either

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are good,}$$

or

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are not good.}$$

Definition (good tableaux)

A standard tableau T is *good* if $T = \mathrm{Q}(w)$ and w is good.

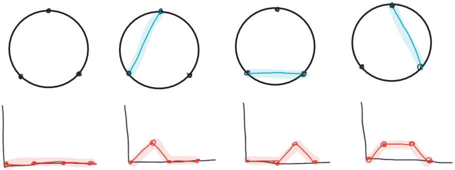
- Question: How many good tableaux are there?

Answer: Good tableaux are new Motzkin objects!

Theorem (2022)

The good standard tableaux, $\{\mathrm{Q}(w) \mid w \in S_n \text{ and } \mathrm{SD}(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

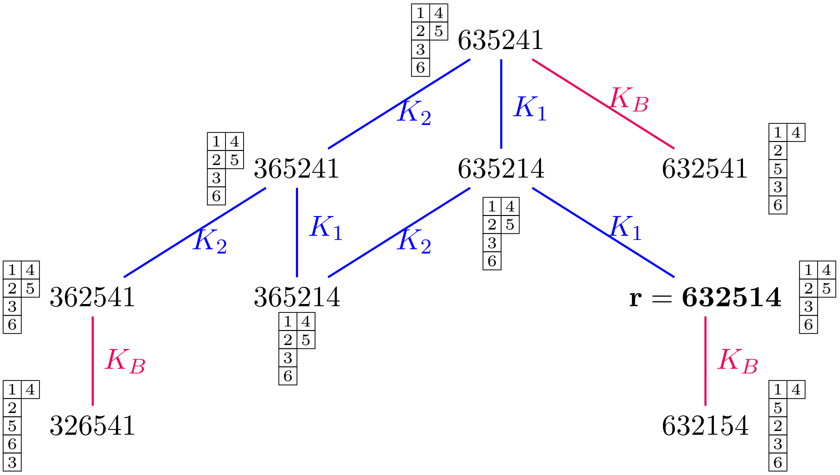
$$M_0 = 1, \qquad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i \, M_{n-2-i}$$



$$M_3 = 4$$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

Question: Characterize permutations with the same soliton decomposition



Permutations connected by *Knuth moves* to $\mathbf{r} = \mathbf{632514}$ and their soliton decompositions

Knuth Relations

Suppose v, $w \in S_n$ and $x < y < z$.

1. v and w differ by a Knuth relation of the **first kind** (K_1) if

$$v = x_1 \ldots yxz \ldots x_n \text{ and } w = x_1 \ldots yzx \ldots x_n \text{ or vice versa}$$

2. v and w differ by a Knuth relation of the **second kind** (K_2) if

$$v = x_1 \ldots xzy \ldots x_n \text{ and } w = x_1 \ldots zxy \ldots x_n \text{ or vice versa}$$

In addition, v and w differ by a Knuth relation of **both kinds** (K_B) if they differ by K_1 and they differ by K_2, that is,

$$v = x_1 \ldots y_1 xzy_2 \ldots x_n \text{ and } w = x_1 \ldots y_1 zxy_2 \ldots x_n \text{ or vice versa}$$

where $x < y_1, y_2 < z$

Example $\mathbf{326}154 \sim^{K_1} \mathbf{362}154$ $\qquad$ $362\mathbf{15}4 \sim^{K_B} 362\mathbf{51}4$

We say that v and w are *Knuth equivalent* if they differ by a finite sequence of Knuth relations.

P-tableaux and Knuth moves

Theorem (Knuth, 1970)

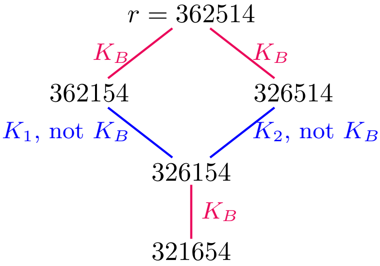
- *There is a path of Knuth moves from w to the row reading word of $P(w)$.*
- *Two permutations have the same P tableau iff they are in the same Knuth equivalence class.*

Example

The Knuth equivalence class of the row word $r = 362514$ of

1	4
2	5
3	6

:



Soliton decompositions and Knuth moves

The soliton decomposition is preserved by non-K_B Knuth moves, but one K_B move changes the soliton decomposition.

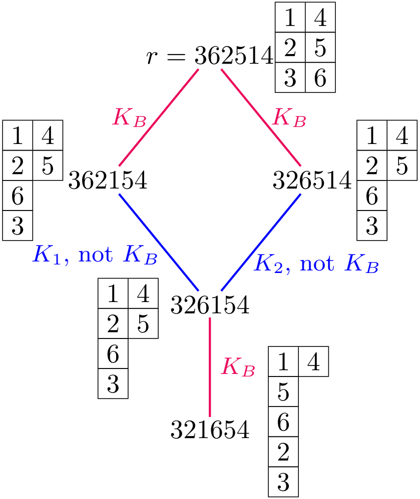
Theorem (2020)

Let r denote the row reading word of $\mathrm{P}(w)$.

- If there exists a path of *non-K_B* Knuth moves from w to r, then $\mathrm{SD}(w) = \mathrm{P}(w)$. In particular, $\mathrm{SD}(r) = \mathrm{P}(r)$.
- If there exists a path from w to r containing an *odd* number of K_B moves, then $\mathrm{SD}(w) \neq \mathrm{P}(w)$.

Example

Soliton decompositions of the Knuth equivalence class of 362154:



Further questions

- Characterize good permutations using consecutive permutation patterns. (Note: this is impossible to do using classical permutation patterns.)
- Define and study continuous box-ball system (on the real line with balls labeled by the real numbers)

Y	O	U	!
A	N	K	
T	H		

Greene's theorem, slide 1/3

Definition (longest k-increasing subsequences)

A subsequence σ of w is called *k-increasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an increasing subsequence of w. Let $\mathrm{i}_k := \mathrm{i}_k(w)$ denote the length of a longest k-increasing subsequence of w.

Example (Let $w = 5623714$.)

- The longest 1-increasing subsequences are
  567, 237, and 234.
- The longest 2-increasing subsequence is given by
  $562374 = 567 \sqcup 234$.
- A longest 3-increasing subsequence (among others) is given by
  $5623714 = 56 \sqcup 237 \sqcup 14$.
- Thus, $\mathrm{i}_1 = 3, \quad \mathrm{i}_2 = 6, \quad$ and $\quad \mathrm{i}_k = 7$ if $k \geq 3$.

Greene's theorem, slide 2/3

Definition (longest k-decreasing subsequences)

Similarly, a subsequence σ of w is called *k-decreasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an decreasing subsequence of w. Let $\mathrm{d}_k := \mathrm{d}_k(w)$ denote the length of a longest k-decreasing subsequence of w.

Example (Let $w = 5623714$.)

- The longest 1-decreasing subsequences are
  521, 621, 531, and 631.
- A longest 2-decreasing subsequence (among others) is given by $52714 = 521 \sqcup 74$.
- A longest 3-decreasing subsequence (among others) is given by $5623714 = 52 \sqcup 631 \sqcup 74$.
- Thus, $\mathrm{d}_1 = 3$, $\mathrm{d}_2 = 5$, and $\mathrm{d}_k = 7$ if $k \geq 3$.

Greene's theorem, slide 3/3

Theorem (Greene, 1974)

Suppose $w \in S_n$. *Let* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ *denote the RS partition of* w, *that is, let* $\lambda = \operatorname{sh} P(w)$. *Let* $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ *denote the conjugate of* λ. *Then, for any* k,

$$\mathrm{i}_k(w) = \lambda_1 + \lambda_2 + \ldots + \lambda_k,$$
$$\mathrm{d}_k(w) = \mu_1 + \mu_2 + \ldots + \mu_k.$$

Example

By Greene's theorem, the RS partition is equal to $\lambda = (\mathrm{i}_1, \mathrm{i}_2 - \mathrm{i}_1, \mathrm{i}_3 - \mathrm{i}_2) = (3, 3, 1)$. We can verify this by computing the RS tableaux

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}, \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

A localized version of Greene's theorem, slide 1/3

Definition (A localized version of longest k-increasing subsequences)

Let $\mathrm{i}(u) :=$ the length of a longest increasing subsequence of u.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{I}_k(w) = \max_{w=u_1|\cdots|u_k} \sum_{j=1}^{k} \mathrm{i}(u_j)$, where the maximum is taken over ways of writing w as a concatenation $u_1 \mid \cdots \mid u_k$ of consecutive subsequences.

Example

Let $w = 5623714$. For short, we write $\mathrm{I}_k := \mathrm{I}_k(w)$. Then

$\mathrm{I}_1 = \mathrm{i}(w) = 3$ (since the longest increasing subsequences are 567, 237, 234),

$\mathrm{I}_2 = 5$ (witnessed by $56|23714$ or $56237|14$),

$\mathrm{I}_3 = 7$ (witnessed uniquely by $56|237|14$), and

$\mathrm{I}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem, slide 2/3

Definition (A localized version of longest k-decreasing subsequences)

Let $\mathrm{D}(u) \coloneqq 1 + |\{\text{descents of } u\}|$.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{D}_k(w) = \max_{w = u_1 \sqcup \cdots \sqcup u_k} \sum_{j=1}^{k} \mathrm{D}(u_j)$, where the maximum is taken over ways to write w as the union of disjoint subsequences u_j of w.

Example

Let $w = 5623714$. For short, we write $\mathrm{D}_k \coloneqq \mathrm{D}_k(w)$. Then

$\mathrm{D}_1 = \mathrm{D}(w) = 1 + |\text{descents of } 5623714| = 1 + |\{2, 5\}| = 3$,

$\mathrm{D}_2 = 6$ (take subsequences 531 and 6274, among other partitions),

$\mathrm{D}_3 = 7$ (take subsequences 52, 631, and 74, among other partitions), and

$\mathrm{D}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem, slide 3/3

Theorem (Lewis–Lyu–Pylyavskyy–Sen 2019)

Suppose $w \in S_n$. Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \ldots)$ denote $\operatorname{sh}\operatorname{SD}(w)$. *Let $M = (M_1, M_2, M_3, \ldots)$ denote the conjugate of Λ. Then, for any k,*

$$\mathrm{I}_k(w) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_k,$$
$$\mathrm{D}_k(w) = M_1 + M_2 + \ldots + M_k.$$

Example

Let $w = 5623714$. By the above theorem, $\operatorname{sh}\operatorname{SD}(w) = (\mathrm{I}_1, \mathrm{I}_2 - \mathrm{I}_1, \mathrm{I}_3 - \mathrm{I}_2) = (3, 2, 2)$. We can verify this by computing the soliton decomposition $\operatorname{SD}(w)$, which turns out to be the (non-standard) tableau

1	3	4
2	7	
5	6	

.

Note: $\operatorname{sh}\operatorname{SD}(w) = (3, 2, 2)$ is smaller than $\operatorname{sh} P(w) = (3, 3, 1)$ in the dominance order.

Examples: permutations with L-shaped SD

A permutation with L-shaped SD which is not a column reading word:

$w = 3217654 = (13)(47)(56)$ is a noncrossing involution.

$$\mathrm{P}(w) = \mathrm{Q}(w) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 \\ \cline{1-1} \end{array} \qquad \text{and} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1} 7 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

An involution which is neither noncrossing nor a column reading word:

$v = 5274163 = (15)(37)$ has a crossing.

$$\mathrm{P}(v) = \mathrm{Q}(v) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 & 7 \\ \cline{1-2} \end{array} \qquad \text{and} \qquad \mathrm{SD}(v) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} 7 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

Permutations connected by K_B moves & have the same SD

Two permutations with the same SD which are connected by K_B moves:

$r = 35124$ $\quad \mathrm{SD}(r) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

K_B

$\mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\quad 31524$

K_2, not K_1

$\mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\quad 31254$

K_2, not K_1

$\mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\quad 13254$

K_B

$w = 13524$ $\quad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$