# From permutations to waves, triangulations, and representations

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# Permutations

Let $S_n$ denote the set of permutations on the numbers $\{1, \ldots, n\}$.

We will represent permutations in two ways,

- in *two-line notation*, as

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ w(1) & w(2) & \ldots & w(n) \end{pmatrix}, \text{ and}$$

- in *one-line notation*, as $w = w(1)w(2)\cdots w(n) \in S_n$.

## Example

A permutation in $S_5$

- in two-line notation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, and
- in one-line notation: 21435

# Part I: Box-ball systems and tableaux

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# Solitary waves (solitons)

**Scott Russell's first encounter (August 1834)**

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped.

[The mass of water in the channel] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, ... and after a chase of one or two miles I lost it in the windings of the channel."

Soliton on the Scott Russell Aqueduct on the Union Canal (July 1995)

(ma.hw.ac.uk/solitons/press.html)

Two soliton animation: www.desmos.com/calculator/86loplpajr

# (Multicolor) box-ball system, Takahashi 1993

A *box-ball system* is a dynamical system of box-ball configurations.

- At each configuration, balls are labeled by numbers 1 through $n$ in an infinite strip of boxes.
- Each box can fit at most one ball.

## Example

A possible box-ball configuration:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\cdots$ | **4** | **5** | **2** | **3** | **6** | **1** | | | | | | | | | $\cdots$ |

# Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 0$ | $\cdots$ | **4** | **5** | **2** | **3** | **6** | **1** | | | | $\cdots$ |
| | $\cdots$ | **4** | **5** | **2** | **3** | **6** | | **1** | | | $\cdots$ |
| | $\cdots$ | **4** | **5** | | **3** | **6** | **2** | **1** | | | $\cdots$ |
| | $\cdots$ | **4** | **5** | | | **6** | **2** | **1** | **3** | | $\cdots$ |
| | $\cdots$ | | **5** | **4** | | **6** | **2** | **1** | **3** | | $\cdots$ |
| | $\cdots$ | | | **4** | **5** | **6** | **2** | **1** | **3** | | $\cdots$ |
| $t = 1$ | $\cdots$ | | | **4** | **5** | | **2** | **1** | **3** | **6** | $\cdots$ |

# Box-ball moves ($t = 0$ through $t = 5$)

| | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 0 \cdots$ | **4** | **5** | **2** | **3** | **6** | **1** | | | | | | | | | | | | | | | | $\cdots$ |
| $t = 1 \cdots$ | | | **4** | **5** | | **2** | **1** | **3** | **6** | | | | | | | | | | | | | $\cdots$ |
| $t = 2 \cdots$ | | | | | **4** | **5** | **2** | | | **1** | **3** | **6** | | | | | | | | | | $\cdots$ |
| $t = 3 \cdots$ | | | | | | | **4** | **2** | **5** | | | | **1** | **3** | **6** | | | | | | | $\cdots$ |
| $t = 4 \cdots$ | | | | | | | | **4** | | **2** | **5** | | | | | **1** | **3** | **6** | | | | $\cdots$ |
| $t = 5 \cdots$ | | | | | | | | | **4** | | | **2** | **5** | | | | | | **1** | **3** | **6** | $\cdots$ |

# Solitons and steady state

## Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future box-ball moves.

## Example

The strings **4**, **25**, and **136** are solitons:

| | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 3$ $\cdots$ | | | | | | | **4** | **2** | **5** | | | | **1** | **3** | **6** | | | | | | | $\cdots$ |
| $t = 4$ $\cdots$ | | | | | | | | **4** | | **2** | **5** | | | | | **1** | **3** | **6** | | | | $\cdots$ |
| $t = 5$ $\cdots$ | | | | | | | | | **4** | | | **2** | **5** | | | | | | **1** | **3** | **6** | $\cdots$ |

After a finite number of box-ball moves, the system reaches a *steady state* where:

- each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

# Question (steady-state time)

The time when a permutation $w$ first reaches steady state is called the *steady-state time* of $w$.

- Find a formula to compute the steady-state time of a permutation, without needing to run box-ball moves.

# Tableaux (English notation)

## Definition

- A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if its rows and columns are increasing.

## Example

Standard Tableaux:

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | |
| 6 | 7 | |

| | | |
|---|---|---|
| 1 | 3 | 6 |
| 2 | 5 | |
| 4 | | |

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 7 | |
| 5 | 8 | |
| 6 | | |

Nonstandard Tableau:

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 5 | 6 | 7 |
| 4 | | |

# Soliton decomposition

## Definition

To construct *soliton decomposition* $\mathrm{SD}(w)$ of $w$, start with the one-line notation of $w$, and run box-ball moves until we reach a steady state; the 1st row of $\mathrm{SD}(w)$ is the rightmost soliton, the 2nd row of $\mathrm{SD}(w)$ is the next rightmost soliton, and so on.

## Example

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t=0$ | $\cdots$ | **4** | **5** | **2** | **3** | **6** | **1** | | | | | | | | | | | | | $\cdots$ |
| $t=1$ | $\cdots$ | | | **4** | **5** | | **2** | **1** | **3** | **6** | | | | | | | | | | $\cdots$ |
| $t=2$ | $\cdots$ | | | | | **4** | **5** | **2** | | | **1** | **3** | **6** | | | | | | | $\cdots$ |
| $t=3$ | $\cdots$ | | | | | | | **4** | **2** | **5** | | | | **1** | **3** | **6** | | | | $\cdots$ |
| $t=4$ | $\cdots$ | | | | | | | | **4** | | **2** | **5** | | | | | **1** | **3** | **6** | $\cdots$ |

$\mathrm{SD}(452361) =$

| | | |
|---|---|---|
| 1 | 3 | 6 |
| 2 | 5 | |
| 4 | | |

with shape $(3, 2, 1)$.

# RSK bijection

The classical Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from $S_n$ onto pairs of size-$n$ standard tableaux of equal shape.

Example

Let $w = \mathbf{452361}$. Then

$$\mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \quad \text{and} \quad \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} .$$

# RSK bijection example

Let $w = \mathbf{452361}$.

$$\text{P}: \quad \mathbf{4} \qquad \begin{array}{cc} 4 & \mathbf{5} \end{array} \qquad \begin{array}{cc} \mathbf{2} & 5 \\ 4 & \end{array} \qquad \begin{array}{cc} 2 & \mathbf{3} \\ 4 & 5 \end{array} \qquad \begin{array}{ccc} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{array} \qquad \begin{array}{ccc} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \qquad \text{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$\text{Q}: \quad \mathbf{1} \qquad \begin{array}{cc} 1 & \mathbf{2} \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \mathbf{3} & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & \mathbf{4} \end{array} \qquad \begin{array}{ccc} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{array} \qquad \text{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

## Insertion and bumping rule for P

- Insert $x$ into the first row of P.
- If $x$ is larger than every element in the first row, add $x$ to the end of the first row.
- If not, replace the smallest number larger than $x$ in row 1 with $x$. Insert this number into the row below following the same rules.

## Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

# Q($w$) determines the box-ball dynamics of $w$

**Theorem (SUMRY 2021)**

If $\mathrm{Q}(v) = \mathrm{Q}(w)$, then

- $v$ and $w$ first reach steady state at the same time, and
- the soliton decompositions of $v$ and $w$ have the same shape

**Example**

$$v = 21435 \text{ and } w = 31425$$

$$\mathrm{Q}(v) = \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

Both $v$ and $w$ have steady-state time $t = 1$

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

# Questions (steady-state time)

Two permutations are said to be *Q-equivalent* if they have the same Q-tableau.

- Given a Q-tableau, find a formula to compute the steady-state time for all permutations in this Q-tableau equivalence class.
- Find an upper bound for steady-state times of all permutations in $S_n$.

# L-shaped soliton decompositions

## Theorem (SUMRY 2021)

If a permutation has an L-shaped soliton decomposition $\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} & \multicolumn{2}{c}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{2}{c}{} \end{array}\cdots$,

then its steady-state time is either $t = 0$ or $t = 1$.

## Remark

Such permutations include “noncrossing involutions” and “column words” of standard tableaux.

## Example

Both $v = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$v = (12)(34)$ and $w = 31425$ is the column word of $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$.

# Maximum steady-state time

**Theorem (UConn 2020)**

If $n \geq 5$ and

$$\mathrm{Q}(w) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline n \\ \cline{1-1}\end{array} \ldots \begin{array}{|c|c|}\hline n-2 & n-1 \\ \hline\end{array},$$

then the steady-state time of $w$ is $n-3$.

**Conjecture**

For $n \geq 4$, the steady-state time of a permutation in $S_n$ is at most $n-3$.

# A permutation with steady-state time $n - 3$

Let $w = 452361 \in S_6$. Then $\mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ and the steady-state time of $w$ is $3 = n - 3$.

| | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 0$ | $\cdots$ | **4** | **5** | **2** | **3** | **6** | **1** | | | | | | | | | | | | | | | | $\cdots$ |
| $t = 1$ | $\cdots$ | | | **4** | **5** | | **2** | **1** | **3** | **6** | | | | | | | | | | | | | $\cdots$ |
| $t = 2$ | $\cdots$ | | | | | **4** | **5** | **2** | | | **1** | **3** | **6** | | | | | | | | | | $\cdots$ |
| $t = 3$ | $\cdots$ | | | | | | | **4** | **2** | **5** | | | | **1** | **3** | **6** | | | | | | | $\cdots$ |
| $t = 4$ | $\cdots$ | | | | | | | | **4** | | **2** | **5** | | | | | **1** | **3** | **6** | | | | $\cdots$ |
| $t = 5$ | $\cdots$ | | | | | | | | | **4** | | | **2** | **5** | | | | | | **1** | **3** | **6** | $\cdots$ |

# Questions (soliton decomposition)

- When is the soliton decomposition SD a standard tableau?
- Characterize the permutations with the same soliton decompositions

# When is SD(w) a standard tableau?

**Example**

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \quad \mathrm{SD}(21435) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \mathrm{SD}(31425) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

**Theorem (UConn 2020 + D. Grinberg)**

Given a permutation $w$, the following are equivalent:

1. $\mathrm{SD}(w)$ is standard
2. $\mathrm{SD}(w) = \mathrm{P}(w)$
3. the shape of $\mathrm{SD}(w)$ is equal to the shape of $\mathrm{P}(w)$

**Definition (good permutations)**

We say that a permutation $w$ is *good* if the tableau $\mathrm{SD}(w)$ is standard.

# Q($w$) determines whether $w$ is good

**Proposition**

Given a standard tableau $T$, either

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are good,}$$

or

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are not good.}$$

**Definition (good tableaux)**

A standard tableau $T$ is *good* if $T = \mathrm{Q}(w)$ and $w$ is good.

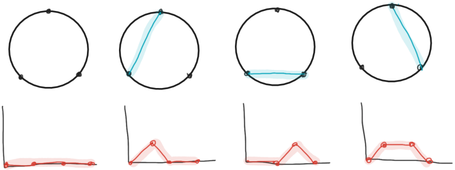
- Question: How many good tableaux are there?

# Answer: Good tableaux are new Motzkin objects!

**Theorem (SUMRY 2022)**

The good standard tableaux, $\{Q(w) \mid w \in S_n \text{ and } SD(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

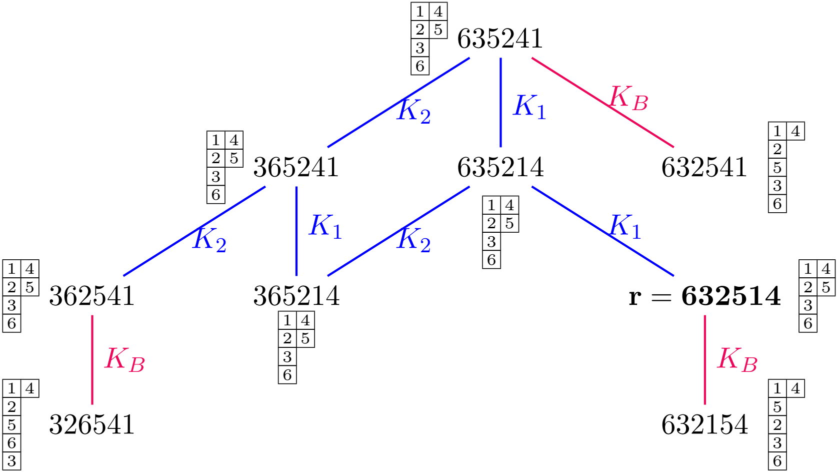
$$M_0 = 1, \qquad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}$$



$$M_3 = 4$$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

# Further question: Characterize permutations with the same soliton decomposition




Permutations connected by *Knuth moves* to $\mathbf{r} = \mathbf{632514}$ and their soliton decompositions

# The end of part I

| | | | |
|---|---|---|---|
| $Y$ | $O$ | $U$ | ! |
| $A$ | $N$ | $K$ | |
| $T$ | $H$ | | |

# Part II: Triangulations and quiver representations

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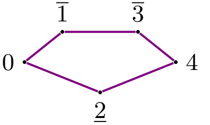
# Inspiration: $\eta$ map (Björner–Wachs 1997, Reading 2004)

A surjection $\eta : S_3 \twoheadrightarrow \{$ triangulations of 

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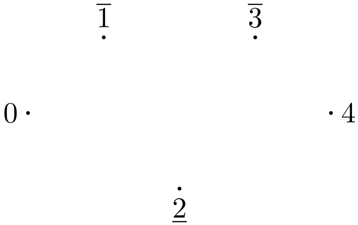
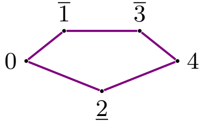


# Inspiration: $\eta$ map (Björner–Wachs 1997, Reading 2004)

A surjection $\eta : S_3 \twoheadrightarrow \{$ triangulations of

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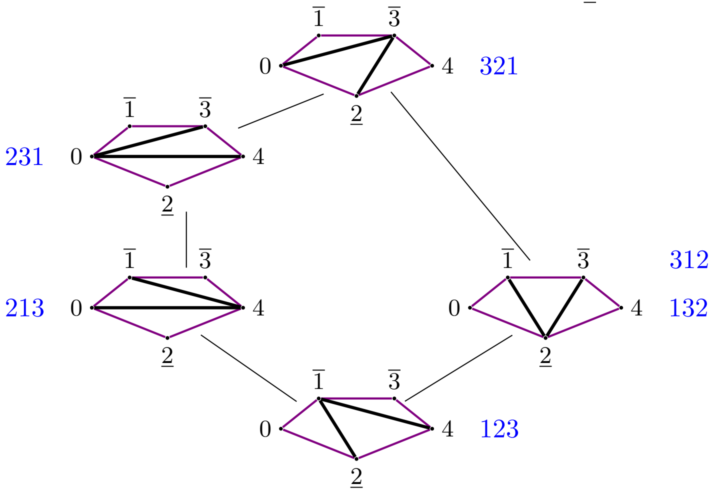


# Inspiration: $\eta$ map (Björner–Wachs 1997, Reading 2004)

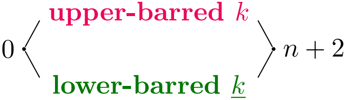
In general, we have a surjection

$$\eta^Q : S_{n+1} \twoheadrightarrow \{\text{triangulations of } P(Q)\}, \qquad \text{where}$$

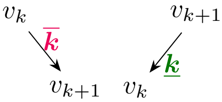
$Q$ is a type $A_{n+2}$ quiver, i.e. an orientation of the Dynkin diagram

$$v_1 \;\text{——}\; v_2 \;\text{——}\; \cdots \;\text{——}\; v_{n+2}\;,$$

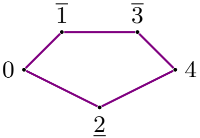
$P(Q)$ is the $(n+3)$-gon with vertices $0, 1, 2, \ldots, n+2$ from left to right



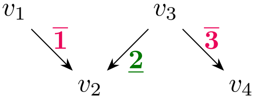

via the rule




Ex: $P(Q) =$




for $Q =$



# Quiver representations

Let $Q$ be a quiver, e.g. $\quad Q = \; v_1 \longrightarrow v_2 \longleftarrow v_3 \longrightarrow v_4$

A *representation* $M$ of a quiver $Q$ is assigning

- a $\mathbb{C}$-vector space to each vertex of $Q$
- a $\mathbb{C}$-linear map to each arrow of $Q$

Ex: $$M = \mathbb{C}^2 \xrightarrow{\begin{bmatrix}1&0\\1&0\\0&0\end{bmatrix}} \mathbb{C}^3 \xleftarrow{\begin{bmatrix}0\\1\\1\end{bmatrix}} \mathbb{C} \xrightarrow{[1]} \mathbb{C}$$

## Proposition (Gabriel 1972)

If $Q$ is a Dynkin quiver of type $A_{n+2}$, the “indecomposable” representations of $Q$ are the representations $M(i,j)$ with $\mathbb{C}$ on each of $v_i, v_{i+1}, \ldots, v_j$, and the identity map on each arrow (with $1 \leq i \leq j \leq n+2$).

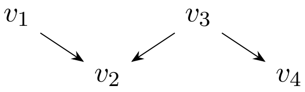
Ex: $$M(2,4) = \begin{smallmatrix} & 3 & \\ 2 & & 4 \end{smallmatrix} = 0 \xrightarrow{[1]} \mathbb{C} \xleftarrow{[1]} \mathbb{C} \xrightarrow{[1]} \mathbb{C}$$

# The Auslander–Reiten quiver

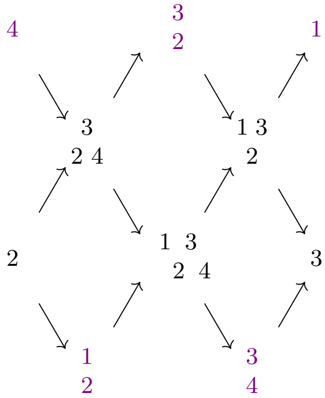
The *Auslander–Reiten quiver* of $Q$ is a directed graph $\Gamma_Q$ with

- vertices: indecomposable representations of $Q$
- arrows: "irreducible" morphisms

Ex: Quiver Q




The Auslander–Reiten quiver of $Q$

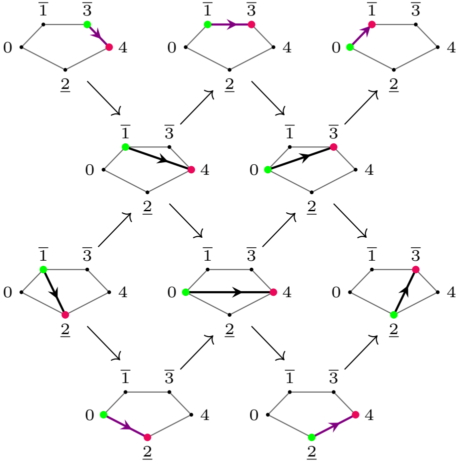


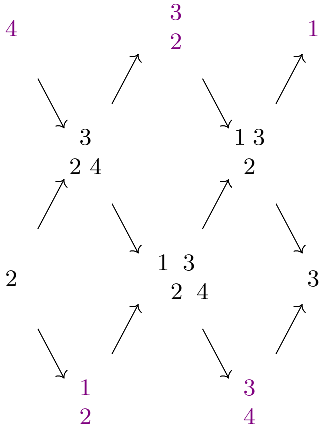
# A model for the AR quiver inspired by the $\eta$ map

## Theorem (Barnard–G.–Meehan–Schiffler 2019)

Line segment from $i$ to $j \longleftrightarrow$ indecomposable representation $M(i+1, j)$

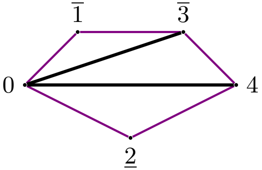
Counterclockwise pivot $\longleftrightarrow$ irreducible morphism

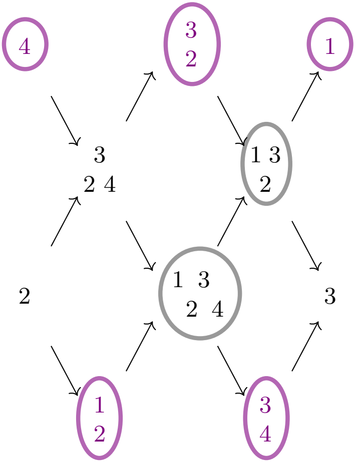


# Question

- Triangulations of $P(Q)$ $\longleftrightarrow$ ??



# A new class of quiver representations

## Definition (BGMS 19)

Let $Q$ be a type $A$ quiver. A representation $T$ of $Q$ is *maximal almost rigid (mar)* if

(1) $T$ has (# of vertices) + (# of arrows) non-isomorphic summands

(2) $T$ is *almost rigid*, that is, for each pair $A, B$ of indecomposable summands of $T$, if $0 \to B \to E \to A \to 0$ is a short exact sequence then $E \cong A \oplus B$ or $E$ is indecomposable.

## Remark

Condition (1) can be replaced with "$T$ is maximal with respect to (2)".

# A new class of Catalan objects

## Theorem (BGMS 19)

Let $Q$ be a type $A$ quiver. Then

$$\{\text{Triangulations of } P(Q)\} \longleftrightarrow \{\text{mar representations of } Q\}$$

## Definition

The *$n$-th Catalan number* is the number of triangulations of the $(n+2)$-gon.

## Corollary

The mar representations are counted by the Catalan numbers.

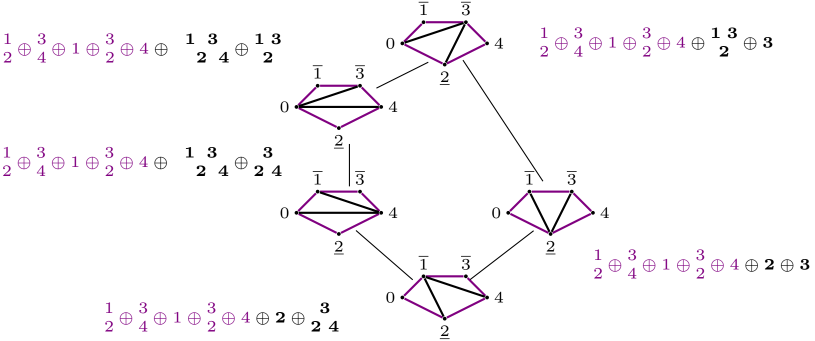
**Further work (with E. Barnard, R. Coelho Simões, and R. Schiffler):**

Tell similar stories about mar objects in the setting of “gentle quivers with relations”, “string quivers with relations”, and more.

# Partial order on the mar representations

## Theorem (BGMS 19)

We put a natural Cambrian poset structure on the mar representations.

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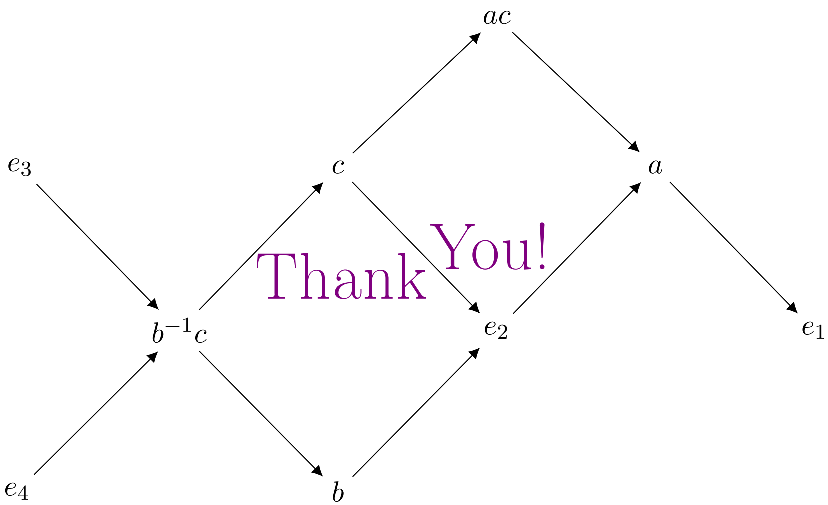

$ac$
$e_3$
$c$
$a$
Thank You!
$b^{-1}c$
$e_2$
$e_1$
$e_4$
$b$