

From permutations to waves, triangulations, and representations

Emily Gunawan, University of Oklahoma

University of Massachusetts Lowell
Mathematics & Statistics
Friday, March 17, 2023

Permutations

Let S_n denote the set of permutations on the numbers $\{1, \dots, n\}$.

We will represent permutations in two ways,

- ▶ in *two-line notation*, as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ w(1) & w(2) & \dots & w(n) \end{pmatrix}, \text{ and}$$

- ▶ in *one-line notation*, as $w = w(1)w(2)\cdots w(n) \in S_n$.

Example

A permutation in S_5

- ▶ in two-line notation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, and
- ▶ in one-line notation: 21435

Part I: Box-ball systems and tableaux

Emily Gunawan, University of Oklahoma,
joint with

B. Drucker, E. Garcia, A. Rumbolt, R. Silver (UConn REU 2020)
M. Cofie, O. Fugikawa, M. Stewart, D. Zeng (SUMRY 2021)
S. Hong, M. Li, R. Okonogi-Neth, M. Sapronov, D. Stevanovich, H.
Weingord (SUMRY 2022)

University of Massachusetts Lowell
Mathematics & Statistics
Friday, March 17, 2023

Solitary waves (solitons)

Scott Russell's first encounter (August 1834)

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped.

[The mass of water in the channel] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, ... and after a chase of one or two miles I lost it in the windings of the channel.”



Soliton on the Scott Russell Aqueduct on the Union Canal (July 1995)

(ma.hw.ac.uk/solitons/press.html)

Two soliton animation: www.desmos.com/calculator/86lop1pajr

(Multicolor) box-ball system, Takahashi 1993

A *box-ball system* is a dynamical system of box-ball configurations.

- ▶ At each configuration, balls are labeled by numbers 1 through n in an infinite strip of boxes.
- ▶ Each box can fit at most one ball.

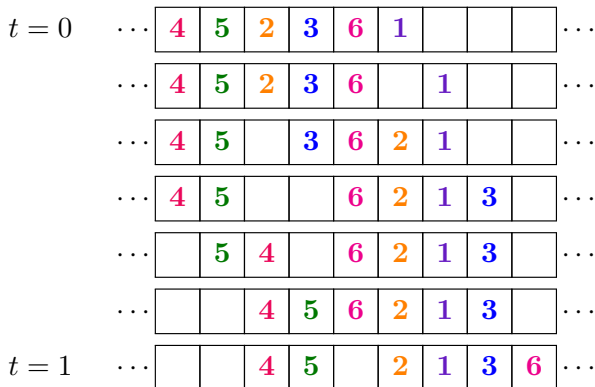
Example

A possible box-ball configuration:

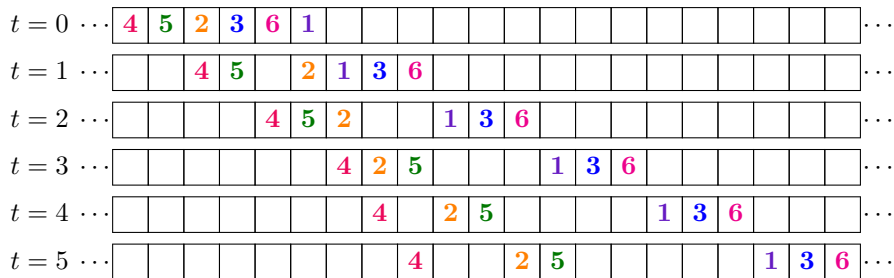


Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.



Box-ball moves ($t = 0$ through $t = 5$)



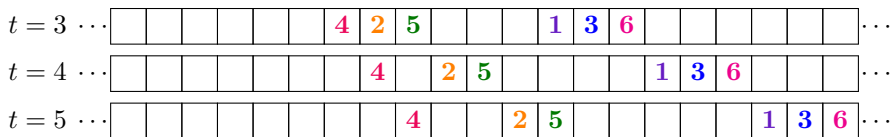
Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future box-ball moves.

Example

The strings **4**, **25**, and **136** are solitons:



After a finite number of box-ball moves, the system reaches a *steady state* where:

- ▶ each ball belongs to one soliton
- ▶ the lengths of the solitons are weakly decreasing from right to left

Question (steady-state time)

The time when a permutation w first reaches steady state is called the *steady-state time* of w .

- ▶ Find a formula to compute the steady-state time of a permutation, without needing to run box-ball moves.

Tableaux (English notation)

Definition

- ▶ A *tableau* is an arrangement of numbers $\{1, 2, \dots, n\}$ into rows whose lengths are weakly decreasing.
- ▶ A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

1	2	4
3	5	
6	7	

1	3	6
2	5	
4		

1	3	4
2	7	
5	8	
6		

Nonstandard Tableau:

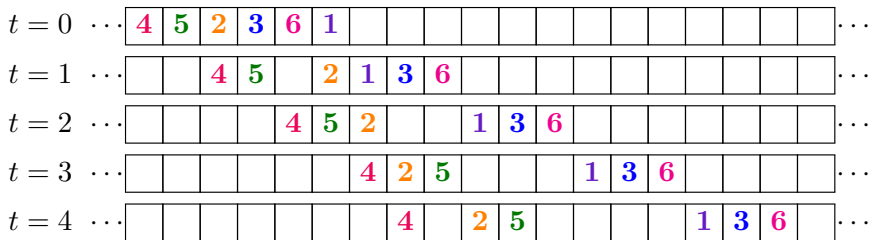
1	2	3
5	6	7
4		

Soliton decomposition

Definition

To construct *soliton decomposition* $SD(w)$ of w , start with the one-line notation of w , and run box-ball moves until we reach a steady state; the 1st row of $SD(w)$ is the rightmost soliton, the 2nd row of $SD(w)$ is the next rightmost soliton, and so on.

Example



$$SD(452361) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} \text{ with shape } (3, 2, 1).$$

RSK bijection

The classical Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (P(w), Q(w))$$

from S_n onto pairs of size- n standard tableaux of equal shape.

Example

Let $w = \mathbf{452361}$. Then

$$P(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad \text{and} \quad Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array}.$$

RSK bijection example

Let $w = 452361$.

P :	4	4	5	2	5	2	3	2	3	6	1	3	6	P(w) =	1	3	6	
			4			4	5			4	5				2	5		
											4							
Q :	1	1	2	1	2	1	2	1	2	5	1	2	5	Q(w) =	1	2	5	
			3			3	4			3	4				3	4		
											6							
															6			

Insertion and bumping rule for P

- ▶ Insert x into the first row of P.
- ▶ If x is larger than every element in the first row, add x to the end of the first row.
- ▶ If not, replace the smallest number larger than x in row 1 with x . Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \dots, n$ in order so that the shape of Q at each step matches the shape of P.

$Q(w)$ determines the box-ball dynamics of w

Theorem (SUMRY 2021)

If $Q(v) = Q(w)$, then

- ▶ v and w first reach steady state at the same time, and
- ▶ the soliton decompositions of v and w have the same shape

Example

$$v = 21435 \text{ and } w = 31425$$

$$Q(v) = Q(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}$$

Both v and w have steady-state time $t = 1$

$$\text{SD}(v) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 4 & & \\ \hline 2 & & \\ \hline \end{array} \quad \text{SD}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 4 & & \\ \hline 3 & & \\ \hline \end{array}$$

Questions (steady-state time)

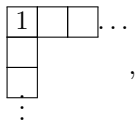
Two permutations are said to be *Q-equivalent* if they have the same Q-tableau.

- ▶ Given a Q-tableau, find a formula to compute the steady-state time for all permutations in this Q-tableau equivalence class.
- ▶ Find an upper bound for steady-state times of all permutations in S_n .

L-shaped soliton decompositions

Theorem (SUMRY 2021)

If a permutation has an L-shaped soliton decomposition



then its steady-state time is either $t = 0$ or $t = 1$.

Remark

Such permutations include “noncrossing involutions” and “column words” of standard tableaux.

Example

Both $v = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$\text{SD}(v) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 4 & & \\ \hline 2 & & \\ \hline \end{array} \quad \text{SD}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 4 & & \\ \hline 3 & & \\ \hline \end{array}$$

$v = (12)(34)$ and $w = 31425$ is the column word of $\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array}$.

Maximum steady-state time

Theorem (UConn 2020)

If $n \geq 5$ and

$$Q(w) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline n & \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline n-2 & n-1 \\ \hline \end{array},$$

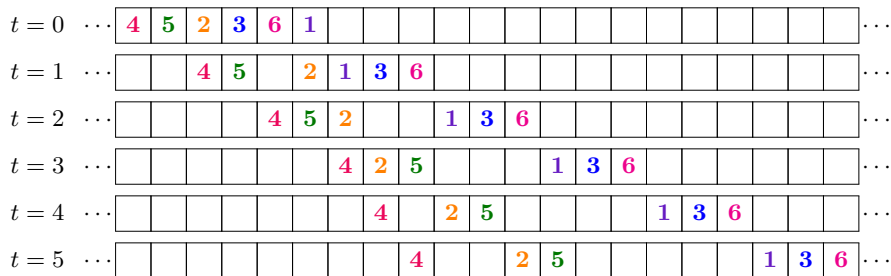
then the steady-state time of w is $n - 3$.

Conjecture

For $n \geq 4$, the steady-state time of a permutation in S_n is at most $n - 3$.

A permutation with steady-state time $n - 3$

Let $w = 452361 \in S_6$. Then $Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array}$ and the steady-state time of w is $3 = n - 3$.



Questions (soliton decomposition)

- ▶ When is the soliton decomposition SD a standard tableau?
- ▶ Characterize the permutations with the same soliton decompositions

When is $SD(w)$ a standard tableau?

Example

$$SD(452361) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad SD(21435) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 4 & & \\ \hline 2 & & \\ \hline \end{array} \quad SD(31425) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 4 & & \\ \hline 3 & & \\ \hline \end{array}$$

Theorem (UConn 2020 + D. Grinberg)

Given a permutation w , the following are equivalent:

1. $SD(w)$ is standard
2. $SD(w) = P(w)$
3. the shape of $SD(w)$ is equal to the shape of $P(w)$

Definition (good permutations)

We say that a permutation w is *good* if the tableau $SD(w)$ is standard.

$Q(w)$ determines whether w is good

Proposition

Given a standard tableau T , either

All w such that $Q(w) = T$ are good,

or

All w such that $Q(w) = T$ are not good.

Definition (good tableaux)

A standard tableau T is *good* if $T = Q(w)$ and w is good.

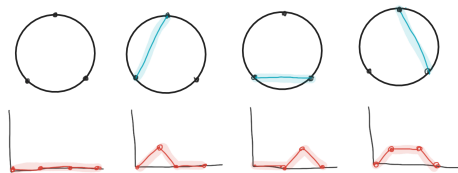
- ▶ Question: How many good tableaux are there?

Answer: Good tableaux are new Motzkin objects!

Theorem (SUMRY 2022)

The good standard tableaux, $\{Q(w) \mid w \in S_n \text{ and } SD(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

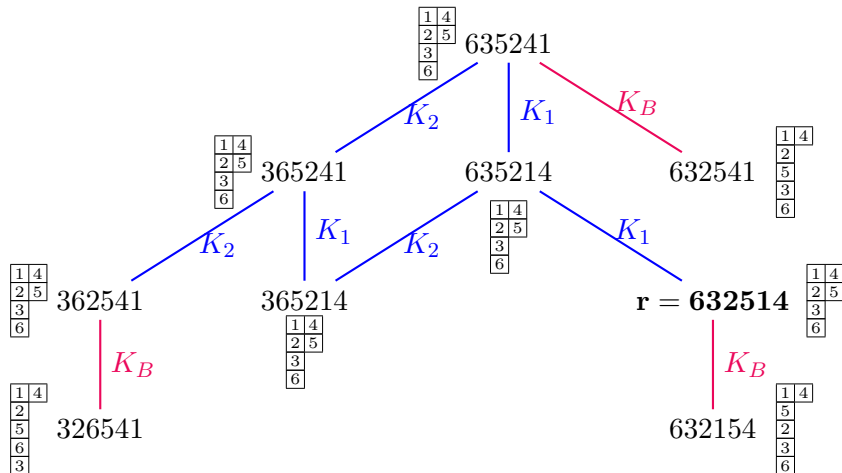
$$M_0 = 1, \quad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}$$



$$M_3 = 4$$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

Further question: Characterize permutations with the same soliton decomposition



Permutations connected by *Knuth moves* to $\mathbf{r} = 632514$ and their soliton decompositions

The end of part I

<i>Y</i>	<i>O</i>	<i>U</i>	!
<i>A</i>	<i>N</i>	<i>K</i>	
<i>T</i>	<i>H</i>		



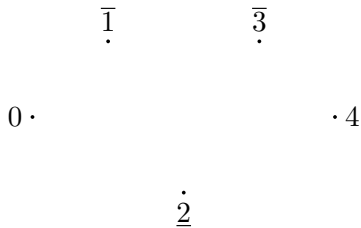
Part II: Triangulations and quiver representations

Emily Gunawan, University of Oklahoma,
joint with
E. Barnard, R. Coelho Simões, E. Meehan, and R. Schiffler

University of Massachusetts Lowell
Mathematics & Statistics
Friday, March 17, 2023

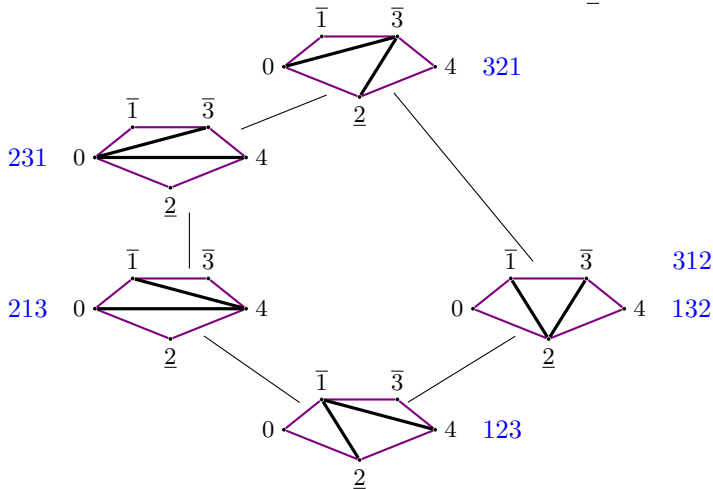
Inspiration: η map (Björner–Wachs 1997, Reading 2004)

A surjection $\eta : S_3 \rightarrow \{ \text{triangulations of } 0 \begin{array}{c} \bar{1} \quad \bar{3} \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \\ \underline{\underline{2}} \end{array} 4 \}$



Inspiration: η map (Björner–Wachs 1997, Reading 2004)

A surjection $\eta : S_3 \twoheadrightarrow \{ \text{triangulations of } 0 \begin{array}{c} \bar{1} \quad \bar{3} \\ \diagdown \quad \diagup \\ \underline{2} \\ \diagup \quad \diagdown \\ 4 \end{array} \}$



Inspiration: η map (Björner–Wachs 1997, Reading 2004)

In general, we have a surjection

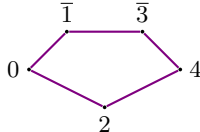
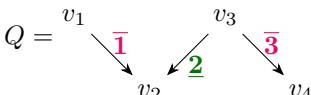
$$\eta^Q : S_{n+1} \rightarrow \{\text{triangulations of } P(Q)\}, \quad \text{where}$$

Q is a type A_{n+2} quiver, i.e. an orientation of the Dynkin diagram

$$v_1 \text{ --- } v_2 \text{ --- } \dots \text{ --- } v_{n+2},$$

$P(Q)$ is the $(n+3)$ -gon with vertices $0, 1, 2, \dots, n+2$ from left to right

$$0 \left\{ \begin{array}{l} \text{upper-barred } \bar{k} \\ \text{lower-barred } \underline{k} \end{array} \right. n+2 \quad \text{via the rule} \quad \begin{array}{cc} v_k & v_{k+1} \\ \searrow \bar{k} & \searrow \underline{k} \\ & v_{k+1} \quad v_k \end{array}$$

Ex: $P(Q) =$  for $Q =$ 

Quiver representations

Let Q be a quiver, e.g. $Q = v_1 \longrightarrow v_2 \longleftarrow v_3 \longrightarrow v_4$

A *representation* M of a quiver Q is assigning

- ▶ a \mathbb{C} -vector space to each vertex of Q
- ▶ a \mathbb{C} -linear map to each arrow of Q

Ex: $M =$

$$\begin{array}{ccccc} \mathbb{C}^2 & & \mathbb{C} & & \mathbb{C} \\ & \searrow & \swarrow & & \searrow \\ & \mathbb{C}^3 & & & \mathbb{C} \end{array}$$

Proposition (Gabriel 1972)

If Q is a Dynkin quiver of type A_{n+2} , the “indecomposable” representations of Q are the representations $M(i, j)$ with \mathbb{C} on each of v_i, v_{i+1}, \dots, v_j , and the identity map on each arrow (with $1 \leq i \leq j \leq n+2$).

Ex: $M(2, 4) =$

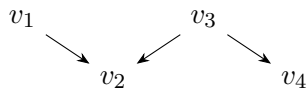
$$\begin{array}{ccccc} 0 & & \mathbb{C} & & \mathbb{C} \\ & \searrow & \swarrow & & \searrow \\ & \mathbb{C} & & & \mathbb{C} \end{array}$$

The Auslander–Reiten quiver

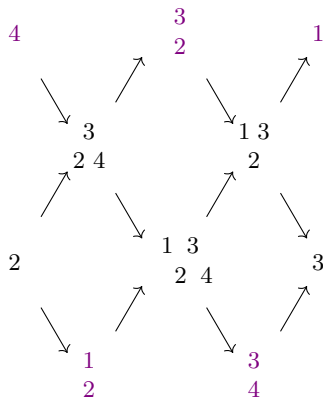
The *Auslander–Reiten quiver* of Q is a directed graph Γ_Q with

- ▶ vertices: indecomposable representations of Q
- ▶ arrows: “irreducible” morphisms

Ex: Quiver Q



The Auslander–Reiten quiver of Q

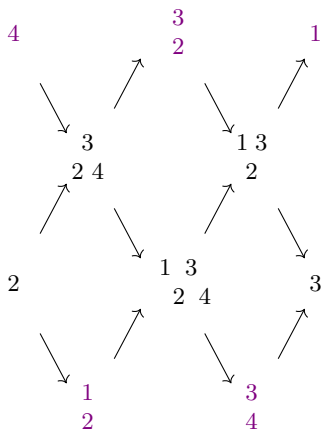
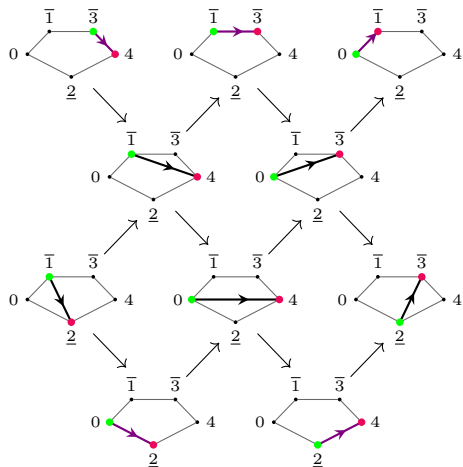


A model for the AR quiver inspired by the η map

Theorem (Barnard–G.–Meehan–Schiffler 2019)

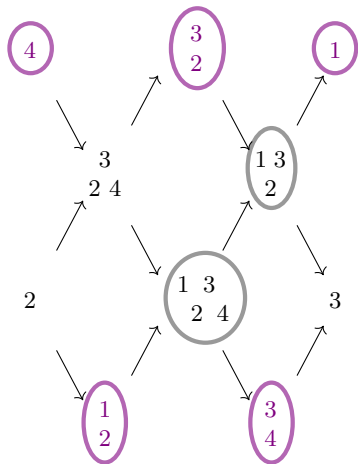
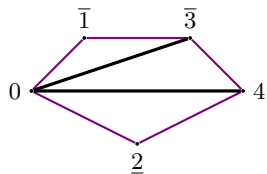
Line segment from i to $j \iff$ indecomposable representation $M(i+1, j)$

Counterclockwise pivot \iff irreducible morphism



Question

► Triangulations of $P(Q) \longleftrightarrow ??$



A new class of quiver representations

Definition (BGMS 19)

Let Q be a type A quiver. A representation T of Q is *maximal almost rigid (mar)* if

- (1) T has $(\# \text{ of vertices}) + (\# \text{ of arrows})$ non-isomorphic summands
- (2) T is *almost rigid*, that is, for each pair A, B of indecomposable summands of T , if $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ is a short exact sequence then $E \cong A \oplus B$ or E is indecomposable.

Remark

Condition (1) can be replaced with “ T is maximal with respect to (2)”.

A new class of Catalan objects

Theorem (BGMS 19)

Let Q be a type A quiver. Then

$$\{\text{Triangulations of } P(Q)\} \longleftrightarrow \{\text{mar representations of } Q\}$$

Definition

The n -th Catalan number is the number of triangulations of the $(n + 2)$ -gon.

Corollary

The mar representations are counted by the Catalan numbers.

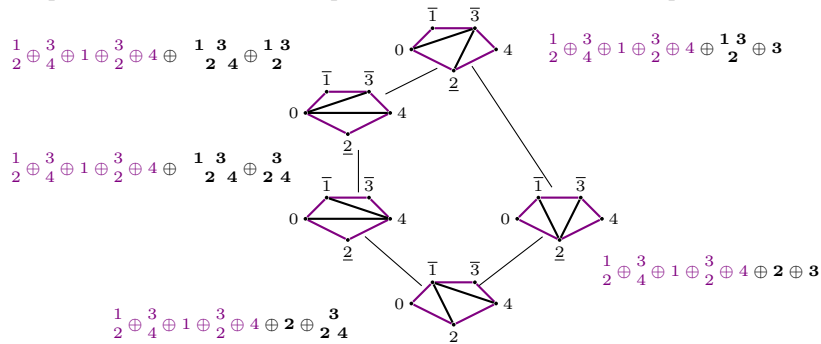
Further work (with E. Barnard, R. Coelho Simões, and R. Schiffler):

Tell similar stories about mar objects in the setting of “gentle quivers with relations”, “string quivers with relations”, and more.

Partial order on the mar representations

Theorem (BGMS 19)

We put a natural Cambrian poset structure on the mar representations.



Further work (with E. Barnard, R. Coelho Simões, and R. Schiffler):

Tell similar stories about mar objects in the setting of “gentle quivers with relations”, “string quivers with relations”, and more.

