Triangulations and maximal almost rigid representations

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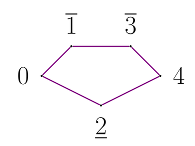
Inspiration: The η map (Björner and Wachs 1997, Reading 2004)

Ex: Consider the surjection

$$\eta : S_3 \twoheadrightarrow \{\text{ triangulations of } \quad \}$$



Take $w = 231 \in S_3$ in one-line notation.

Draw paths

$w(3) = \bar{1}$	0	$\bar{1}$		$\bar{3}$	4
$w(2) = \bar{3}$	0			$\bar{3}$	4
$w(1) = \underline{2}$	0		$\times$		4
Along bottom edges	0		$\underline{2}$		4

Definition [Rea06]: In general, we have a surjection

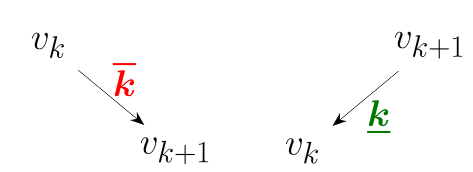
$$\eta^Q : S_{n+1} \twoheadrightarrow \{\text{triangulations of } P(Q)\},$$

where Q is a type A_{n+2} quiver, that is, an orientation of the type A_{n+2} Dynkin diagram

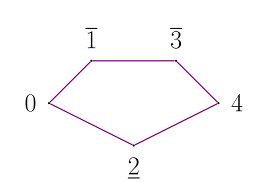
$$v_1 \;—\; v_2 \;—\; \cdots \;—\; v_{n+2},$$

and $P(Q)$ is the $(n+3)$-gon with vertices $0, 1, 2, \ldots, n+2$ drawn from left to right

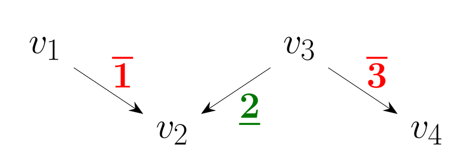
0 ⟨ upper-barred vertices $\bar{k}$ / lower-barred vertices $\underline{k}$ ⟩ $n+2$ via the rule



Ex: $P(Q) =$



for $Q =$

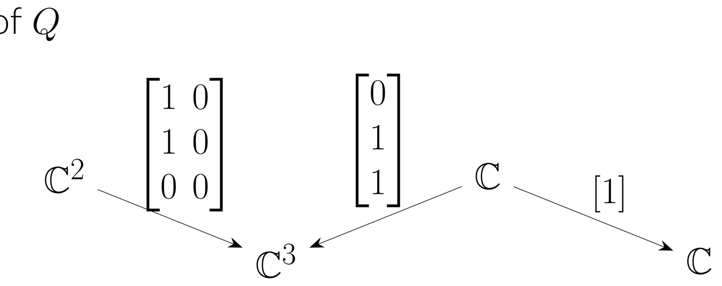


Quiver representations

A *representation* M of a quiver Q is assigning

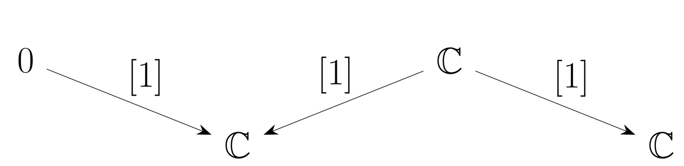
- a $\mathbb{C}$-vector space to each vertex of Q
- a $\mathbb{C}$-linear map to each arrow of Q

Ex: $M =$



Proposition [Gab72]: If Q is a Dynkin quiver of type A_{n+2}, the "indecomposable" representations of Q are the representations $M(i,j)$ with $\mathbb{C}$ on each of $v_i, v_{i+1}, \ldots, v_j$, and the identity map on each arrow (with $1 \le i \le j \le n+2$).

Ex: $M(2,4) = \begin{smallmatrix} & 3 & \\ 2 & & 4 \end{smallmatrix} =$

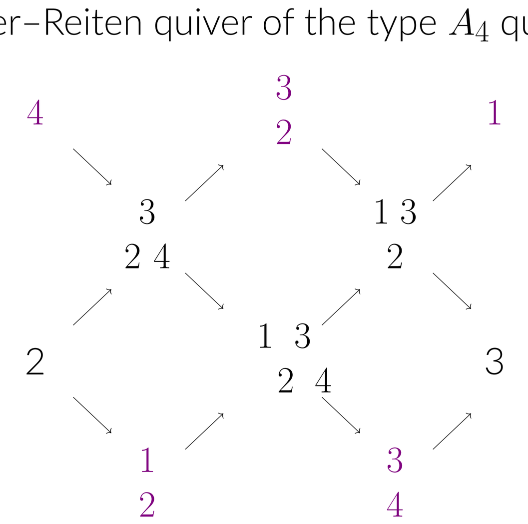


The Auslander–Reiten quiver

The *Auslander–Reiten quiver* of Q is a directed graph Γ_Q with

- vertices: indecomposable representations of Q
- arrows: "irreducible" morphisms

Ex: The Auslander–Reiten quiver of the type A_4 quiver Q above is

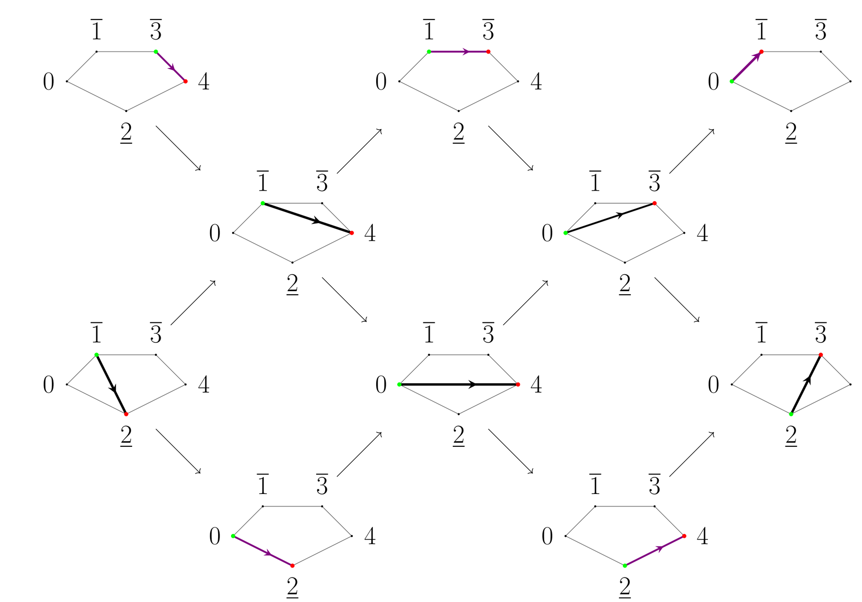


A model for the Auslander–Reiten quiver inspired by the η map

Theorem 1 (Barnard–G.–Meehan–Schiffler [BGMS23])

Line segment from i to j, where $0 \le i < j \le n+1 \longleftrightarrow$ indecomposable representation $M(i+1, j)$

Moving one endpoint counterclockwise $\longleftrightarrow$ irreducible morphism



Question: What is the representation corresponding to a triangulation?

A new class of quiver representations

Classical Definition: Let Q be a type A quiver. A representation T of Q is *maximal rigid* if

(1) T has (# of vertices) non-isomorphic summands

(2) T is *rigid*, that is, for each pair A, B of indecomposable summands of T, if $0 \to B \to E \to A \to 0$ is a short exact sequence then $E \cong A \oplus B$.

Definition 2 [BGMS23]: Let Q be a type A quiver. A representation T of Q is *maximal almost rigid (mar)* if

(1) T has (# of vertices) + (# of arrows) non-isomorphic summands

(2) T is *almost rigid*, that is, for each pair A, B of indecomposable summands of T, if $0 \to B \to E \to A \to 0$ is a short exact sequence then $E \cong A \oplus B$ or E is indecomposable.

Remark: Condition (1) can be replaced with "T is maximal with respect to (2)".

A new class of Catalan objects

Theorem 3 [BGMS23]: Let Q be a type A quiver. Then

$$\{\text{Triangulations of } P(Q)\} \longleftrightarrow \{\text{mar representations of } Q\}$$

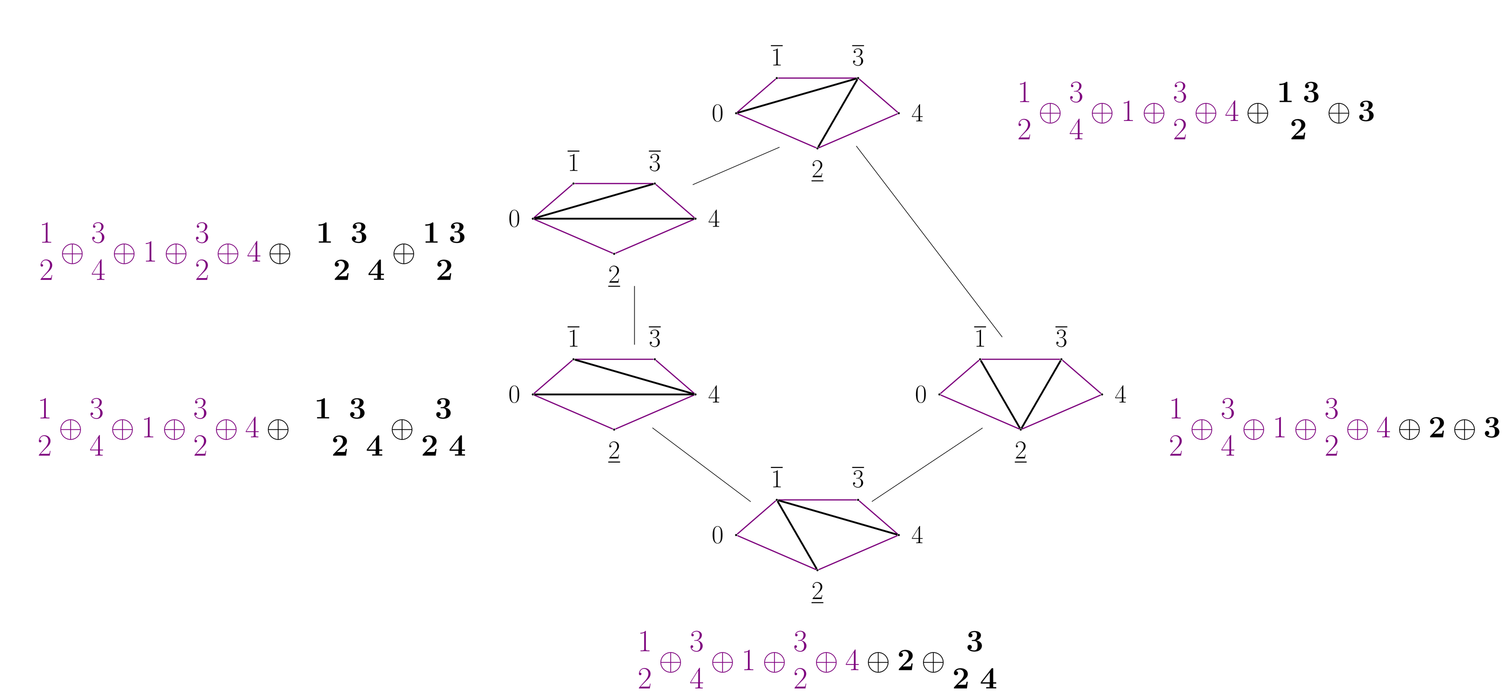
The *n-th Catalan number* is the number of triangulations of the $(n+2)$-gon.

Corollary: The mar representations are counted by the Catalan numbers. ☺

Partial order on the mar representations

We define a partial order on the MAR modules by the cover relation $T_1 < T_2$ iff they differ by exactly one indecomposable summand $M_1 \sim M_2$, and there is a morphism from M_1 to M_2.

Theorem 4 [BGMS23]: The Hasse diagram of this poset is equal to the oriented exchange graph of a smaller type A cluster algebra, and this poset is isomorphic to a Tamari or Cambrian lattice.



Gentle bound quivers

A quiver Q with *relations* R is called *gentle* if

(G1) For each vertex k of Q, there are at most two arrows starting at k and at most two arrows ending at k

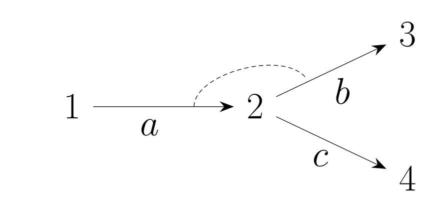
(G2) R is a set of paths of length 2

(G3) For each arrow a of Q, there are at most one arrow b such that $ba \notin R$, and there are at most one arrow c such that $ac \notin R$

(G4) For each arrow a of Q, there are at most one arrow b' such that $b'a \in R$, and there are at most one arrow c' such that $ac' \in R$

(G5) If Q has an oriented cycle, then it must go through a path in R

Ex: The quiver $Q =$

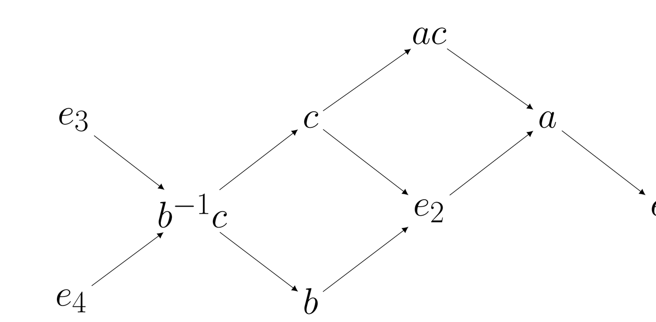


bound by the relation $ab = 0$ is gentle.

A *representation* of a gentle quiver with relations (Q, R) is assigning

- a $\mathbb{C}$-vector space to each vertex of Q
- a $\mathbb{C}$-linear map to each arrow of Q such that the composition of maps along a path in R is the zero map.

Ex: The Auslander–Reiten quiver of the gentle (Q, R) above is



The Auslander–Reiten quiver of a gentle quiver with relations (Q, R) is modeled by a tiled surface [BC21]. Instead of line segments of a polygon, the indecomposable representations correspond to "permissible arcs" of a marked surface.

Theorem 5 (Barnard-Coelho Simões-G.-Schiffler [BCGS]): Let (Q, R) be a gentle quiver with relations. Then

$$\{\text{"Permissible ideal" triangulations of a marked surface}\} \longleftarrow \{\text{mar representations of } (Q, R)\}$$

We define an oriented flip graph of these mar representations which we conjecture to be connected and acyclic in general.

Ex: the mar representations of the gentle (Q, R) above form a chain poset.

$$3 \oplus 4 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\4\end{smallmatrix} \oplus 1 \oplus 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$$

$$3 \oplus 4 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\4\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}2\\4\end{smallmatrix} \oplus 2$$

$$3 \oplus 4 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\4\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}2\\4\end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 4\end{smallmatrix}$$

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