Triangulations & maximal almost rigid modules

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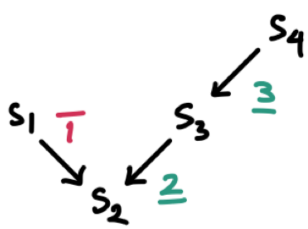
Inspiration: Type A η map (Björner – Wachs 1997, Reading 2004)

Q quiver of type A_{n+2}

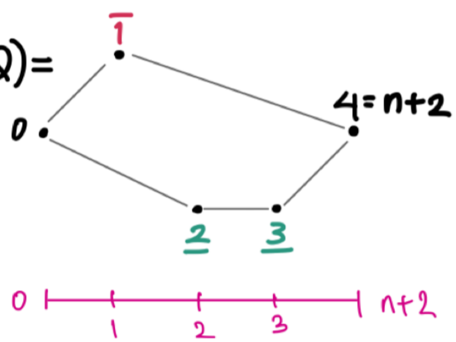
$s_1 - s_2 - \cdots - s_{n+2}$

$Q =$ 

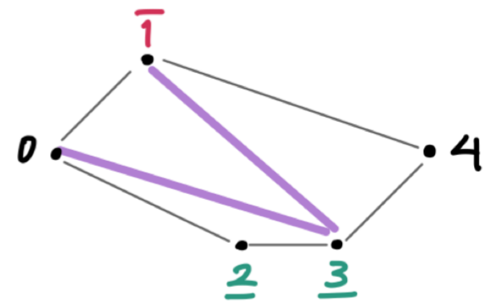
Type A_{2+2}

if $s_i \searrow s_{i+1}$ in Q, let $\bar{i}$ be up

if $s_i \swarrow s_{i+1}$ in Q, let $\underline{i}$ be down

$P(Q) =$ 

$\eta_Q : \underbrace{S_{n+1}}_{\text{symmetric group}} \twoheadrightarrow \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } P(Q) \end{array} \right\}$

$w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3 \longmapsto$ 

along top edges

paths from 0 to n+2

λ_3 $\quad 0 \; \bar{1} \; \underset{\text{remove } 3}{\times} \; 4$ $\quad w(3) = \underline{3}$

λ_2 $\quad 0 \; \bar{1} \; \underline{3} \; 4$ (insert $\bar{1}$) $\quad w(2) = \bar{1}$

λ_1 $\quad 0 \; \underset{\text{remove } 2}{\times} \; \underline{3} \; 4$ $\quad w(1) = \underline{2}$

λ_0 $\quad 0 \; \underline{2} \; \underline{3} \; 4$

along bottom edges

Path algebra

Q quiver e.g. $Q = 1 \xrightarrow{a} 2 \xleftarrow{b} 3 \xleftarrow{c} 4$, $Q_0 := \{\text{vertices}\} = \{1,2,3,4\}$, $Q_1 := \{\text{arrows}\} = \{a, b, c\}$, $\mathbb{k} := \mathbb{C}$

<u>Def</u> The path algebra $\mathbb{k}Q$

- basis: $\{$all paths in Q, including the <u>lazy path</u> e_i at each vertex $i\}$ $= \left\{\begin{array}{llll} e_1, & a, & & \\ & e_2, & & \\ & b, & e_3, & \\ & cb, & c, & e_4 \end{array}\right\}$
- multiplication on two basis elements: $pp' = \begin{cases} \overset{\text{Concatenation}}{pp'} & \text{if } pp' \text{ is a path} \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{k}Q \cong$ algebra of matrices of the form $\begin{bmatrix} \lambda_{e_1} & \lambda_a & 0 & 0 \\ 0 & \lambda_{e_2} & 0 & 0 \\ 0 & \lambda_b & \lambda_{e_3} & 0 \\ 0 & \lambda_{cb} & \lambda_c & \lambda_{e_4} \end{bmatrix}$.

each $\lambda_p \in \mathbb{k}$ is the coefficient of the path p.

Entry in row i, col j $\longleftrightarrow$ path from vertex i to vertex j.

E.g.

$$\begin{array}{c} \\ \\ \\ \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & & 3 & \end{array} \\ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&\lambda_c&0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ 3 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & 2 & & \end{array} \\ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&\lambda_b&0&0\\0&0&0&0 \end{bmatrix} \end{array} = \begin{array}{c} \\ \\ \\ \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & 2 & & \end{array} \\ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&\lambda_{cb}&0&0 \end{bmatrix} \end{array}$$

path c path b path cb

and

$$\begin{array}{c} \\ \\ \\ 3 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & 2 & & \end{array} \\ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&\lambda_b&0&0\\0&0&0&0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & & 3 & \end{array} \\ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&\lambda_c&0 \end{bmatrix} \end{array} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

path b path c 0

Type A module category

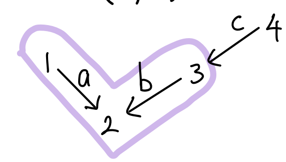
ind Q = Indecomposable modules of the path algebra $\mathbb{K}Q$

$\longleftrightarrow$ intervals $M(i,j)$, $i \leq j$ called "strings"

$\longleftrightarrow$ positive roots in a type A root system

E.g.

$M(1,3)$



string ab^{-1} (or $b\bar{a}^{1}$ or ${}^{1}{}_{2}{}^{3}$)

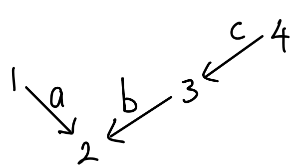
$\alpha_1 + \alpha_2 + \alpha_3$

The Auslander – Reiten quiver of Q is a directed graph Γ_Q with

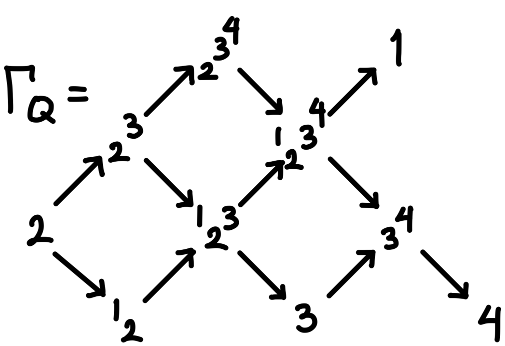
vertices = indecomposable modules

arrows = irreducible morphisms

$Q =$



$\Gamma_Q =$



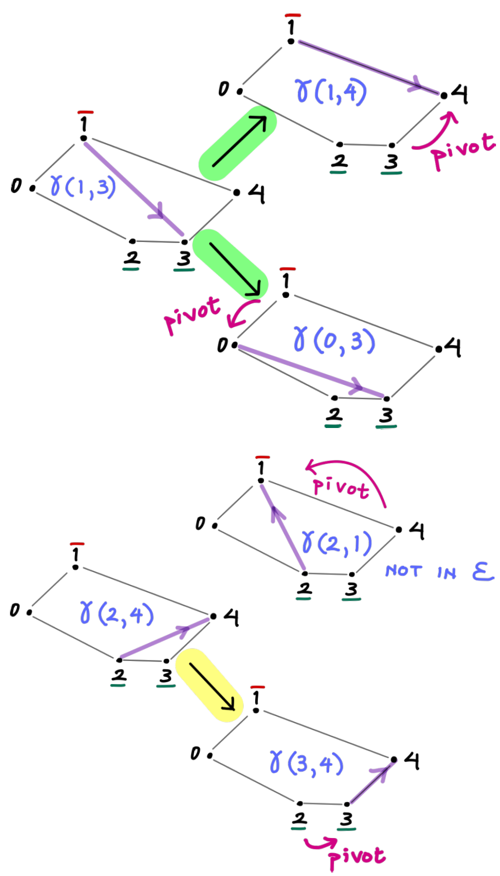
Barnard — G. — Meehan — Schiffler, 2019 [BGMS 19]

A model for ind Q inspired by the η map combinatorics (for type A)

Line segments $\gamma(i,j)$, $0 \leq i < j \leq n+1$ $\longleftrightarrow$ indecomposable modules $M(i+1, j)$
(incl. boundary segments)

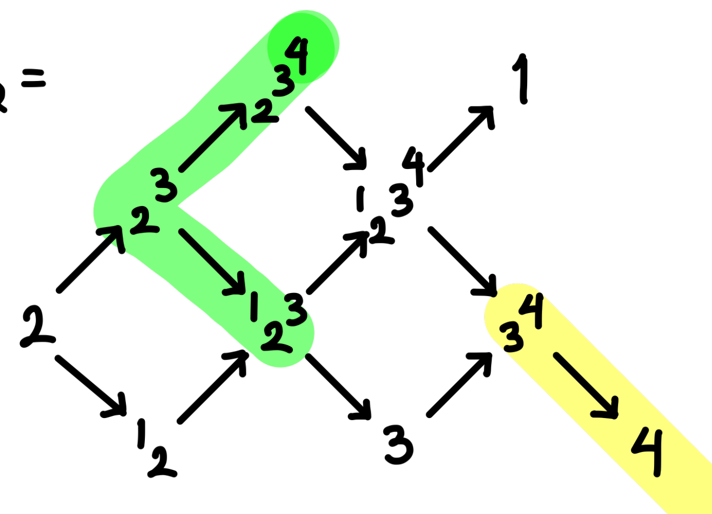
Lower (upper) boundary line segments $\longleftrightarrow$ maximal increasing (decreasing) paths

Pivots $\longleftrightarrow$ irreducible morphisms

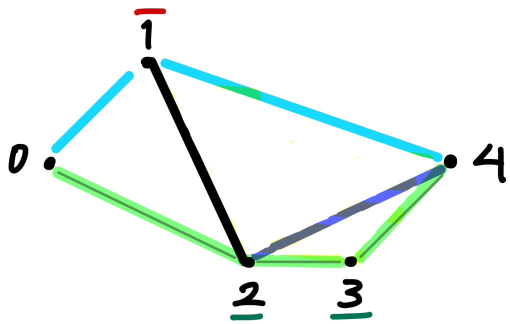


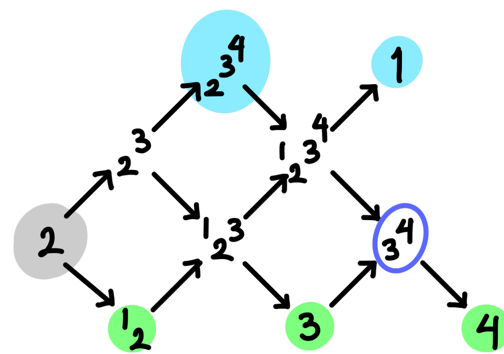
$$Q = 1 \xrightarrow{a} 2 \xleftarrow{b} 3 \xleftarrow{c} 4$$

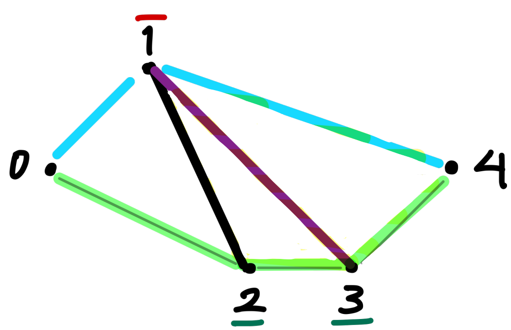
$\Gamma_Q =$

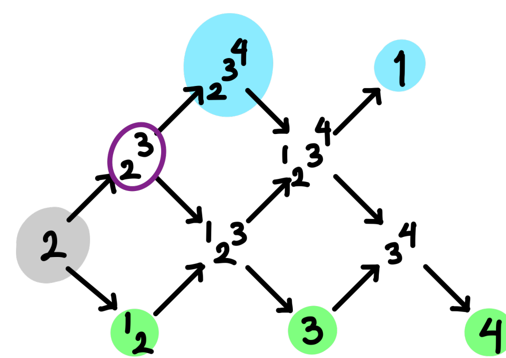


Triangulations $\longleftrightarrow$??









$\mathbb{k}Q$: path algebra of type A

<u>Classical def</u> $T \in \mathrm{mod}(\mathbb{k}Q)$ is <u>maximal rigid</u> if

(T1) T has $|Q_0|$ non-isomorphic summands
# of vertices of Q

(T2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$

called <u>rigid</u>

<u>Def</u> [BGMS 19]

$T \in \mathrm{mod}(\mathbb{k}Q)$ is <u>maximal almost rigid</u> (mar) if

(M1) T has $|Q_0|$ + $|Q_1|$ non-isomorphic summands
# of vertices of Q # of arrows of Q

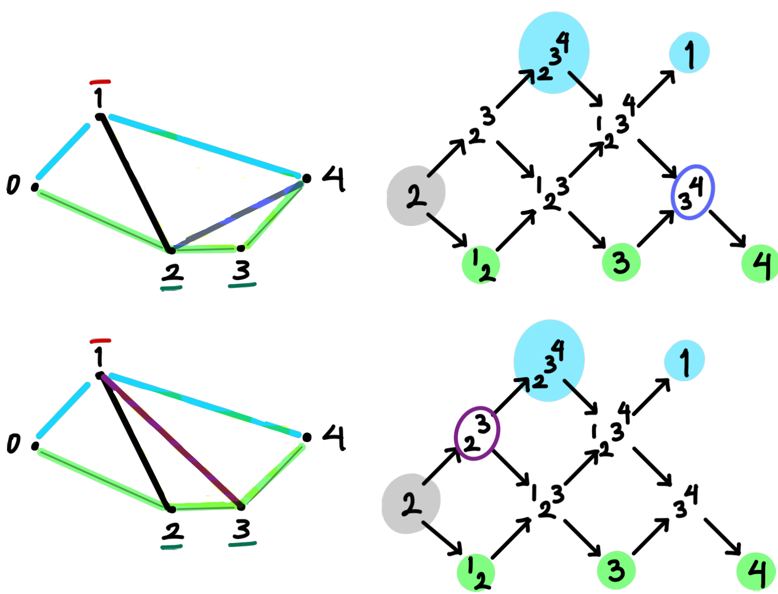
(M2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence,
then $E \cong B \oplus A$ or E is indecomposable

called <u>almost rigid</u>

<u>Rem</u> (M1) can be replaced with :
"T is maximal with respect to (M2)"

Thm 1 [BGMS 19]

$$\left\{\begin{array}{l}\text{triangulations of } P(Q) \\ \text{including boundary edges}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{mar modules} \\ \mathrm{mar}(\Bbbk Q)\end{array}\right\}$$



Corollary

The mar modules (type A) are Catalan objects.

Thm 2 [BGMS 19]

Construct a bigger type A path algebra $\Bbbk\overline{Q}$ where $|\overline{Q}_0| = |Q_0| + |Q_1|$.

Then $M \in \mathrm{mar}(Q) \Rightarrow$

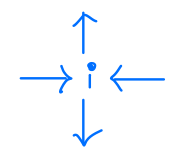
$$\mathrm{End}_A(M) \cong \mathrm{End}_{\overline{A}}(T)$$

where T is tilting in $\mathrm{mod}(\Bbbk\overline{Q})$.

Gentle algebras

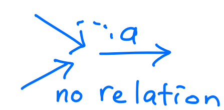
__Def__ A finite-dimensional algebra $A = \mathbb{k}Q/I$ is __gentle__ if :

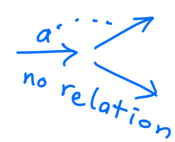
(G1) $\forall$ vertex i of Q, $\exists$ at most 2 arrows starting at i
$\exists$ at most 2 arrows ending at i



(G2) I is generated by paths of length 2

(G3) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b s.t $ba \notin I$
$\exists$ at most 1 arrow c s.t $ac \notin I$





(G4) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b' s.t $b'a \in I$
$\exists$ at most 1 arrow c' s.t $ac' \in I$

__Ex__

$Q = $
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$\mathbb{k}Q$ __Not__ gentle

$Q = $
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

$\mathbb{k}Q/I$ is gentle

$Q = $
$$\begin{tikzcd} & 4 \arrow[dl, "c"'] & \\ 3 & 2 \arrow[l, "d"'] \arrow[r, "a"'] & 1 \arrow[ul, "b"'] \end{tikzcd}$$

$I = \langle ab, ca \rangle$

$\mathbb{k}Q/I$ is also gentle

String modules

A <u>string</u> ω is a walk along the arrows in $Q_1 \cup \underbrace{Q_1^{-1}}_{\{\text{opposite arrow } \alpha^{-1} \mid \alpha \in Q_1\}}$

with • no backtrack aa^{-1} or $a^{-1}a$

• no subwalk v with $v \in I$ or $v^{-1} \in I$
(no going through relations)

<u>Ex</u>

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$$I = \langle ab \rangle$$

• ac and $b^{-1}c$ are strings, ab is <u>not</u> a string

$$\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad \begin{tikzcd} & 3 \\ 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & 4 \end{tikzcd} \qquad \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] & \end{tikzcd}$$

• e_1, e_2, e_3, e_4 are <u>trivial strings</u>

[Butler – Ringel 1987]

• ω string $\longleftrightarrow$ $M(\omega)$ string module

• If $\mathbb{k}Q/I$ is a <u>finite representation type</u> gentle algebra,
finitely many indecomposable modules
indecomposable modules are string modules.

<u>Recall def</u> $T \in \mathrm{mod}(A)$ is <u>tilting</u> if

(T1) T has $|Q_0|$ non-isomorphic summands
\# of vertices of Q

(T2) For each pair A, B of summands of T,

if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$

(T3) T has projective dimension at most 1.

<u>Def</u> $A = \mathbb{k}Q/I$ finite representation type gentle algebra.

$T \in \mathrm{mod}(A)$ is <u>maximal almost rigid</u> (mar) if

(M1) T has $|Q_0|$ + $|Q_1|$ non-isomorphic summands
\# of vertices of Q \# of arrows of Q

(M2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence,
then $E \cong B \oplus A$ or E is indecomposable

called <u>almost rigid</u>

<u>Rem</u> (M1) can be replaced with :
"T is maximal with respect to (M2)"

[Opper – Plamondon – Schroll 2018]
[Baur – Coelho Simões 2018] + extra marked points $*$

$A = \Bbbk Q/I$ is a gentle algebra iff

A comes from a 4-tuple (S, M, P, M^*)

arcs in (S,M)

oriented surface

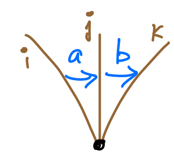
marked points in ∂S

marked points in S

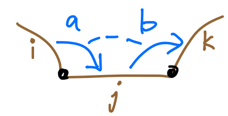
$\bullet \in M$

$* \in M^*$

"Rules":

$i \xrightarrow{a} j \xrightarrow{b} k$ $\quad ab \notin I$ corresponds to configuration 

of P

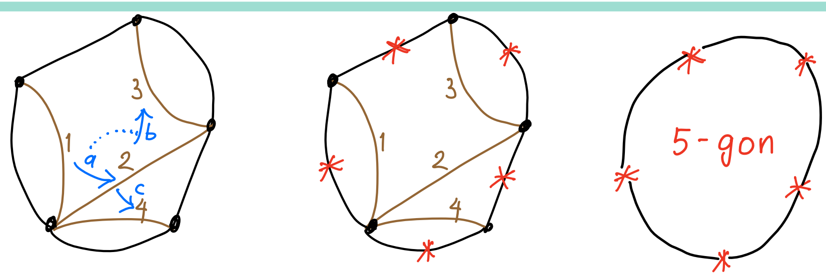
$i \xrightarrow{a} j \xrightarrow{b} k$ $\quad ab \in I$ corresponds to configuration 

of P

Ex

$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$, $\quad 2 \xrightarrow{c} 4$

$I = \langle ab \rangle$



$(S, M, P) \Rightarrow (S, M, P, M^*) \Rightarrow (S, M^*)$

Thm 3 (Barnard – Coelho Simões – G. – Schiffler [B.CS.G.S])

Let $A = \Bbbk Q/I$ be a finite representation type gentle algebra.

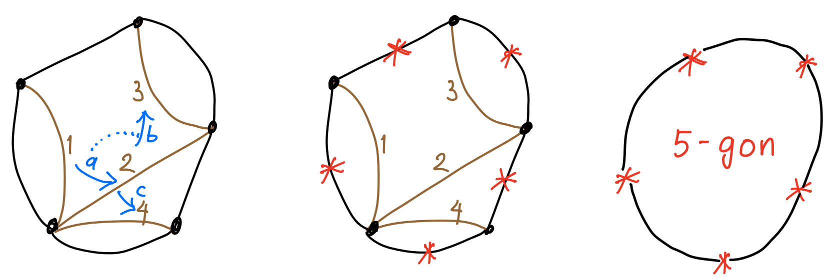
Then $\left\{ \begin{array}{l} \text{"permissible" ideal triangulations of } (S, M^*) \\ \text{including boundary edges} \end{array} \right\} \longleftrightarrow \underbrace{\{\text{mar modules}\}}_{\text{mar}(A)}$

Ex

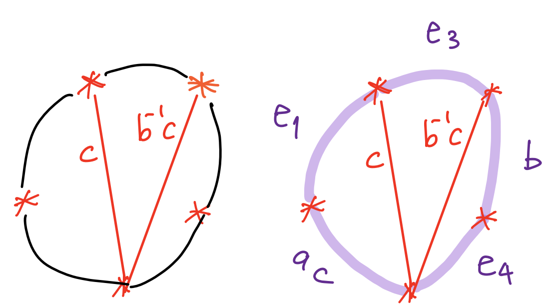
$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

Recall Property (M1):
Each T in mar(A)
has $\underset{4}{|Q_0|} \underset{+}{+} \underset{3}{|Q_1|}$ summands



$(S, M, P) \Rightarrow (S, M, P, M^*) \Rightarrow (S, M^*)$



$c \oplus b'c \oplus \underbrace{e_4 \oplus b \oplus e_3 \oplus e_1 \oplus a_c}_{} \in \text{mar}(A)$

These 5 summands
are required in
every $T \in \text{mar}(A)$

Thm 4

[B.CS.G.S]

Construct a new gentle algebra

$$\bar{A} := \mathbb{k}\bar{Q}/\bar{I} \quad \text{where} \quad |\bar{Q}_0| = |Q_0| + |Q_1| .$$

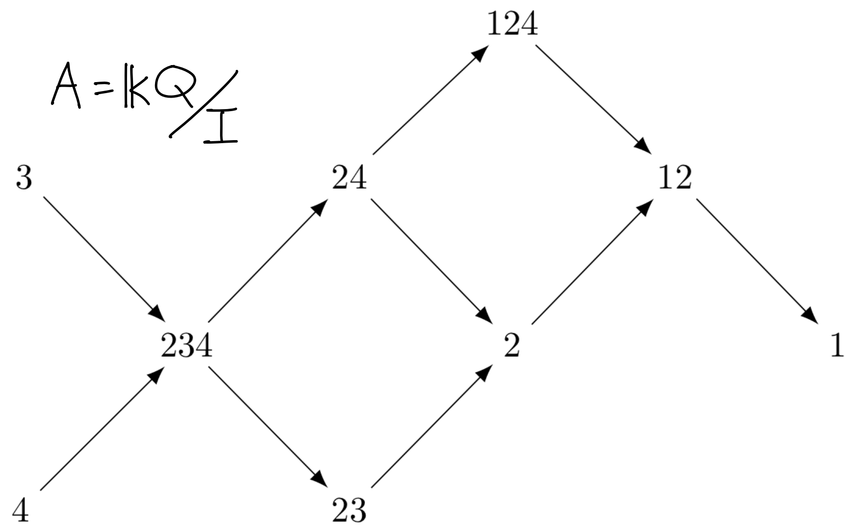
Then $M \in \mathrm{mar}(A) \Rightarrow \mathrm{End}_A(M) \cong \mathrm{End}_{\bar{A}}(T)$

where T is tilting in $\mathrm{mod}(\bar{A})$.

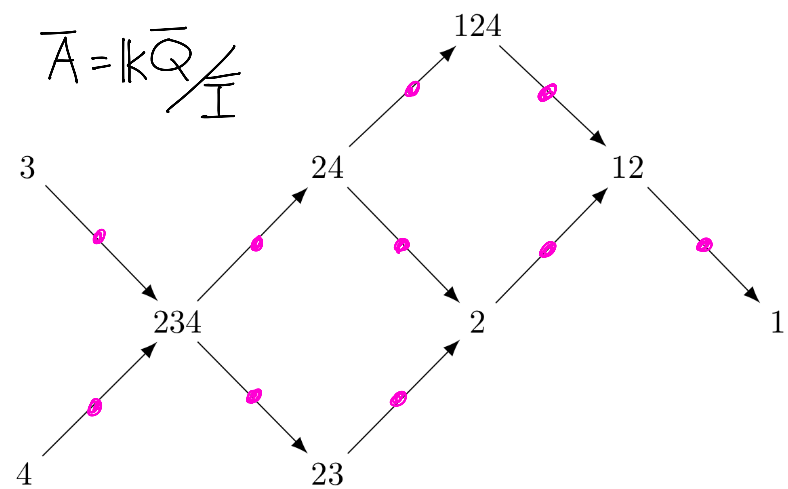
$$Q = \begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[dr, "c"'] & 3 \\ & & 4 \end{tikzcd} \qquad I = \langle ab \rangle$$

$$\bar{Q} = \begin{tikzcd} 1 \arrow[r, "a_1"] & a \arrow[r, "a_2"] & 2 \arrow[r, "b_1"] \arrow[dr] & b \arrow[r] & 3 \\ & & & c \arrow[r] & 4 \end{tikzcd} \qquad \bar{I} = \langle a_2 b_1 \rangle$$

$A = \mathbb{k}Q/I$



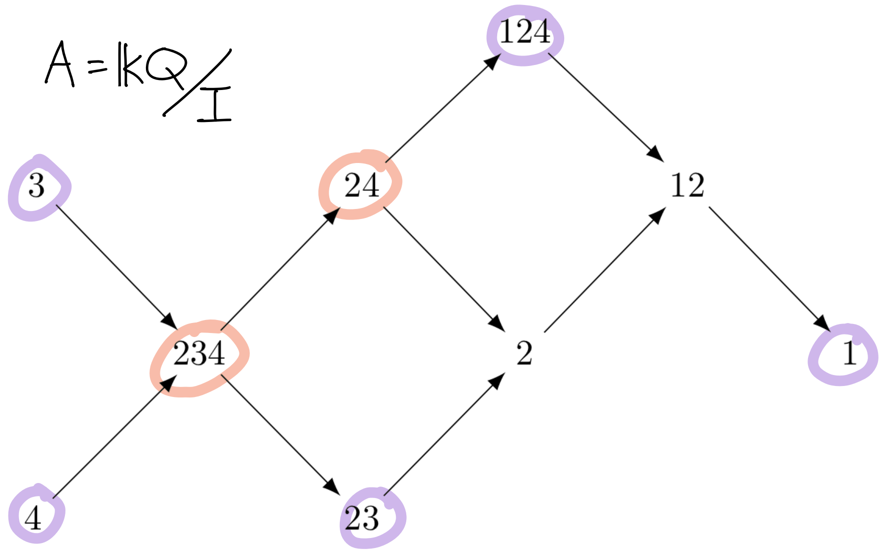
$\bar{A} = \mathbb{k}\bar{Q}/\bar{I}$



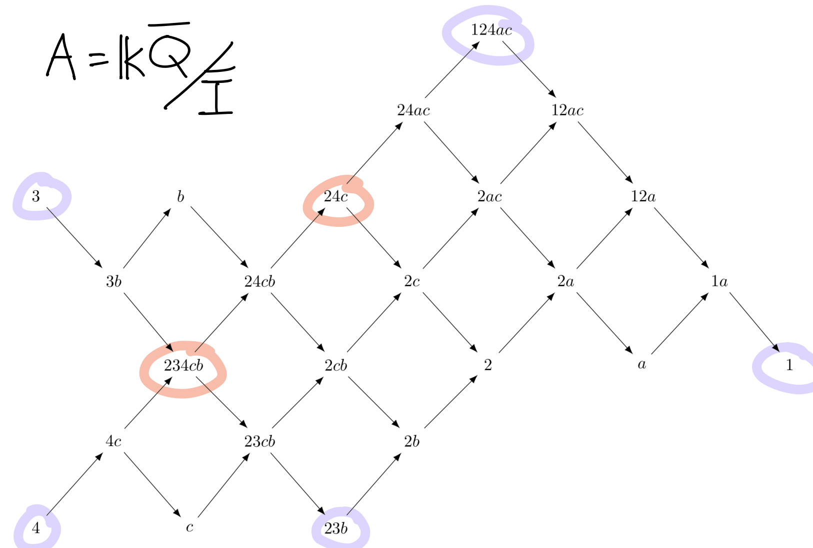
$$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3, \quad 2 \xrightarrow{c} 4 \qquad I = \langle ab \rangle$$

$$\bar{Q} = 1 \xrightarrow{a_1} a \xrightarrow{a_2} 2 \xrightarrow{b_1} b \rightarrow 3, \quad 2 \rightarrow c \rightarrow 4 \qquad \bar{I} = \langle a_2 b_1 \rangle$$

$$A = \mathbb{k}Q/I \qquad \bar{A} = \mathbb{k}\bar{Q}/\bar{I}$$



$M \in \mathrm{mar}(A)$

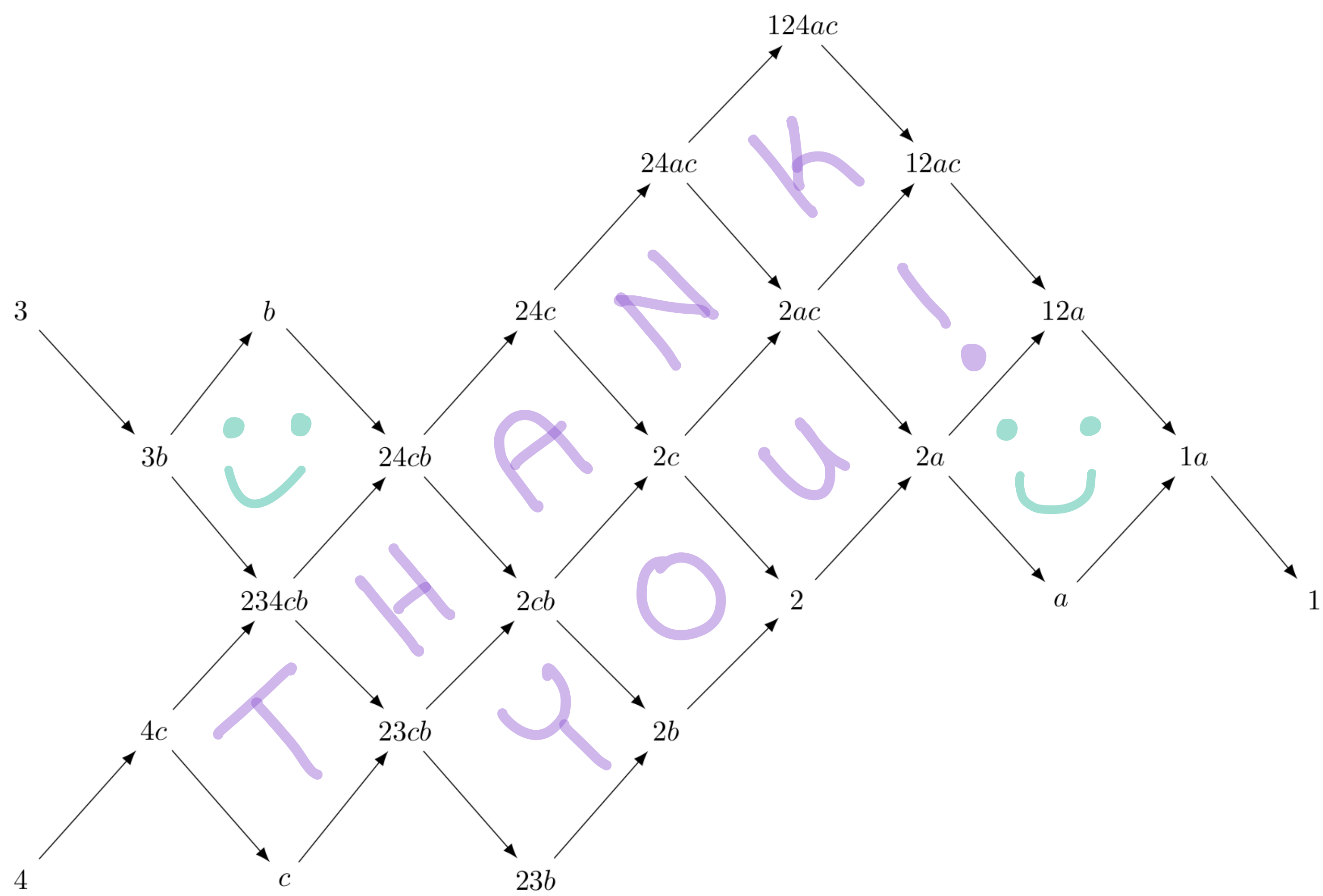


T is a tilting module over $\bar{A}$

Thm 4 $\quad \mathrm{End}_A(M) \cong \mathrm{End}_{\bar{A}}(T)$

mar over A

tilting over $\bar{A}$

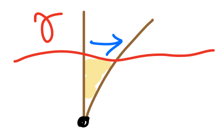

124ac
24ac
12ac
3
b
24c
2ac
12a
3b
24cb
2c
2a
1a
234cb
2cb
2
a
1
4c
23cb
2b
4
c
23b
THANK YOU!

What is a permissible arc? (extra)

string modules of $\Bbbk Q/I$ $\overset{1-1}{\longleftrightarrow}$ <u>permissible</u> arcs γ in S:

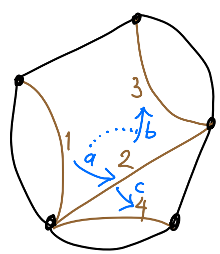
(i) endpoints are in R

(ii) each pair of consecutive crossings of γ and P corresponds to an arrow of Q



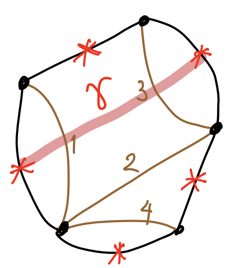
(locally cuts up a triangle)

Ex (S, M, P)

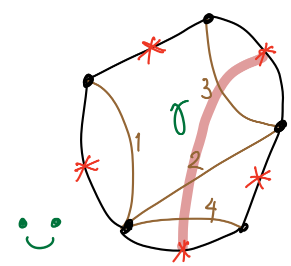


$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

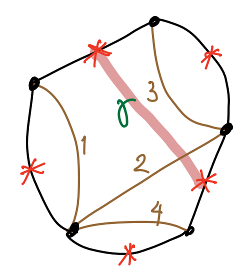


- γ is not permissible
- Consecutive crossings arc 1, arc 3 do not correspond to an arrow of Q



permissible γ

$\leftrightarrow$ string $b^{-1}c$ or $c^{-1}b$

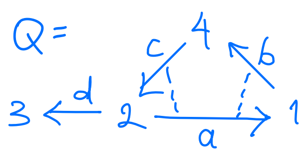


permissible γ

$\leftrightarrow$ trivial string e_2

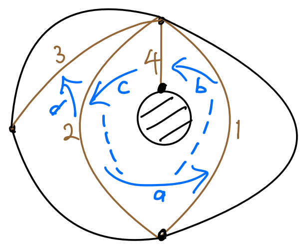
Example (extra)

Q =

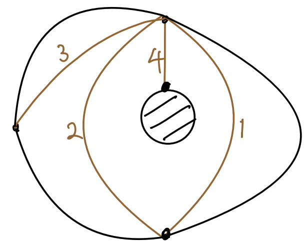


$I = \langle ab, ca \rangle$

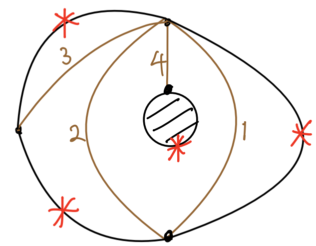
(S, M, P)



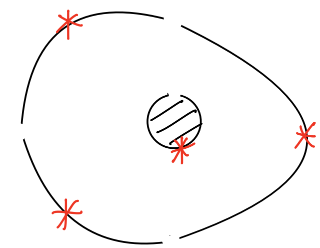
(S, M, P)



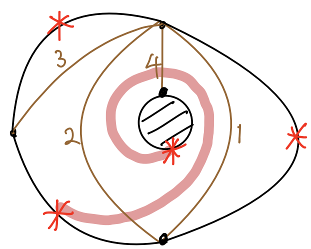
(S, M, P, R)



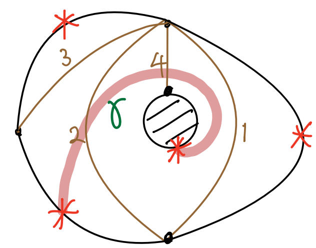
(S, R)



An annulus with
3 points on one bdry
& 1 point on the other



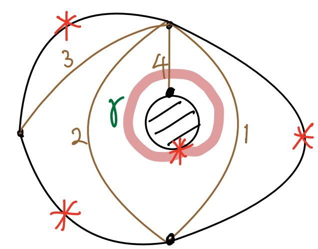
Not permissible
because
the crossings
w/ P at
arc 2, arc 4
do not
correspond
to an arrow in Q



permissible

$\gamma \longleftrightarrow$

string $\begin{array}{c} 2 \\ \uparrow c \\ 4 \end{array}$



permissible

$\gamma \longleftrightarrow$

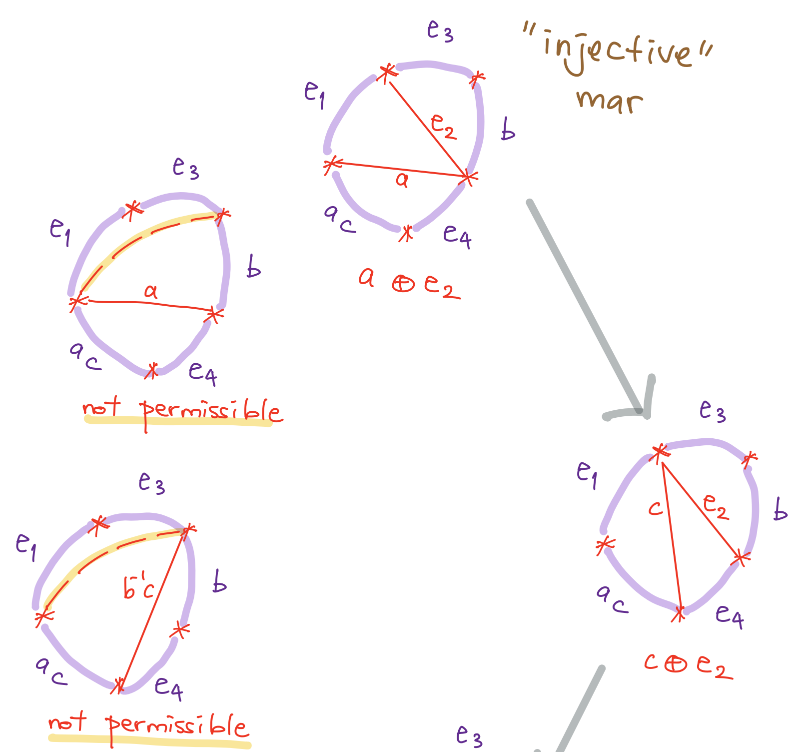
trivial
string e_4

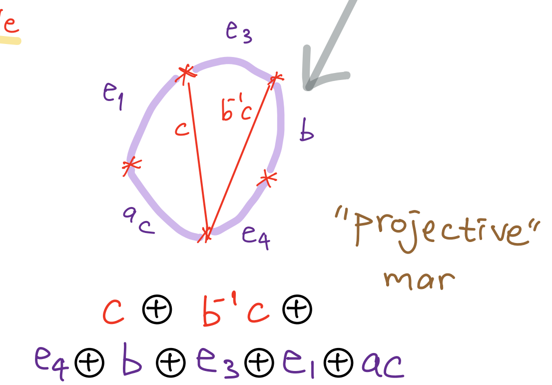
Def/Prop

The oriented flip graph of $\operatorname{mar}(\Bbbk Q/I)$:
arrows are "positive" diagonal flips

Ex The flip graph of $\operatorname{mar}(\Bbbk Q/I)$

for $Q = $
$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$
$I = \langle ab \rangle$ is ...





[Barnard - G. - Meehan - Schiffler 2019]
In type A, the flip graph of $\operatorname{mar}(\Bbbk Q)$ is an oriented exchange graph of a type A cluster algebra & a Tamari/Cambrian lattice