Part I: Box-ball systems and Robinson–Schensted–Knuth tableaux

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Solitary waves (solitons)

Scott Russell's first encounter of solitary waves at the Union Canal:

'I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently **without change of form or diminution of speed**. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.'

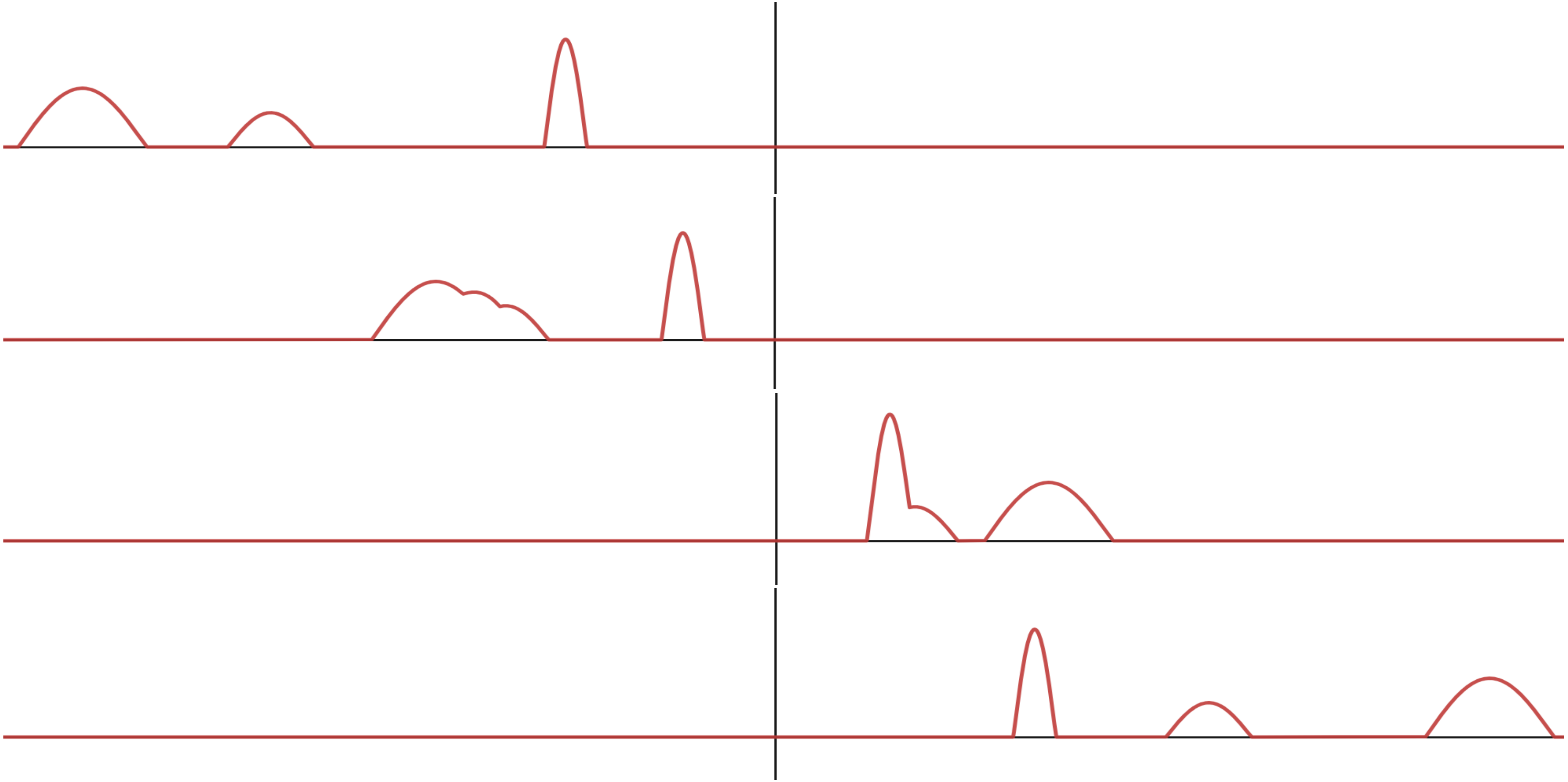
Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, July 1995

Credit:

ma.hw.ac.uk/solitons/press.html

Solitary waves

(Desmos link by D. Zeng)



Multicolor box-ball system (BBS), Takahashi 1993

A *box-ball system* (BBS) is a dynamical system of BBS configurations.

- At each configuration, balls are labeled by numbers 1 through n in an infinite strip of boxes.
- Each box can fit at most one ball.

Example

A possible BBS configuration:

$\cdots$	4	5	2	3	6	1									$\cdots$

Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.

$t = 0$	$\cdots$	4	5	2	3	6	1				$\cdots$
	$\cdots$	4	5	2	3	6		1			$\cdots$
	$\cdots$	4	5		3	6	2	1			$\cdots$
	$\cdots$	4	5			6	2	1	3		$\cdots$
	$\cdots$		5	4		6	2	1	3		$\cdots$
	$\cdots$			4	5	6	2	1	3		$\cdots$
$t = 1$	$\cdots$			4	5		2	1	3	6	$\cdots$

Box-ball moves ($t = 0$ through $t = 5$)

$t = 0$	$\cdots$	4	5	2	3	6	1																$\cdots$
$t = 1$	$\cdots$			4	5		2	1	3	6													$\cdots$
$t = 2$	$\cdots$					4	5	2			1	3	6										$\cdots$
$t = 3$	$\cdots$							4	2	5				1	3	6							$\cdots$
$t = 4$	$\cdots$								4		2	5					1	3	6				$\cdots$
$t = 5$	$\cdots$									4			2	5						1	3	6	$\cdots$

Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future BBS moves.

Example

The strings 4, 25, and 136 are solitons:

$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

After a finite number of BBS moves, the system reaches a *steady state* where:

- the system is decomposed into solitons, i.e., each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

Tableaux (English notation)

Definition

- A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

Nonstandard Tableau:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 6 & 7 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Soliton decomposition

Definition

- Let S_n be the symmetric group on n elements. Represent permutations of S_n in *one-line notation* as
$$w = w(1)w(2)\cdots w(n), \text{ e.g. } w = 452361.$$
- To construct *soliton decomposition* $\mathrm{SD}(w)$ of w, start with the one-line notation of w, and run BBS moves until we rearch a steady state; the 1st row of $\mathrm{SD}(w)$ is the rightmost soliton, the 2nd row of $\mathrm{SD}(w)$ is the next rightmost soliton, and so on.

Example

$t = 3$

						4	2	5				1	3	6						

$t = 4$

							4		2	5					1	3	6			

$\mathrm{SD}(452361) =$

1	3	6
2	5	
4		

with shape $(3, 2, 1)$.

RSK bijection

The Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $w = \mathbf{452361}$. Then $\mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ and $\mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$.

RSK bijection example

Let $w = \mathbf{452361}$.

$$\mathrm{P}: \quad \mathbf{4} \qquad 4\ \mathbf{5} \qquad \begin{matrix} \mathbf{2} & 5 \\ 4 & \end{matrix} \qquad \begin{matrix} 2 & \mathbf{3} \\ 4 & 5 \end{matrix} \qquad \begin{matrix} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{matrix} \qquad \begin{matrix} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{matrix} \qquad \mathrm{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$\mathrm{Q}: \quad \mathbf{1} \qquad 1\ \mathbf{2} \qquad \begin{matrix} 1 & 2 \\ \mathbf{3} & \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 3 & \mathbf{4} \end{matrix} \qquad \begin{matrix} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{matrix} \qquad \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{matrix} \qquad \mathrm{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Insertion and bumping rule for P

- Insert x into the first row of P.
- If x is larger than every element in the first row, add x to the end of the first row.
- If not, replace the smallest number larger than x in row 1 with x. Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

The Q tableau determines the dynamics of a box-ball system

Theorem (SUMRY 2021)

If $\mathrm{Q}(v) = \mathrm{Q}(w)$, then the box-ball systems of v and w are identical if we ignore the ball labels, in particular:

- v and w first reach steady state at the same time, and
- the soliton decompositions of v and w have the same shape

Example

$$v = 21435 \text{ and } w = 31425$$

$$\mathrm{Q}(v) = \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

Both v and w first reach steady state at $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Questions (steady-state time)

The time when a permutation w first reaches steady state is called the *steady-state time* of w.

- Given a Q-tableau, find a formula to compute the steady-state time for all permutations in the Q-tableau class.
- Find an upper bound for steady-state time.

L-shaped soliton decompositions

Theorem (SUMRY 2021)

If a permutation has an L-shaped soliton decomposition $\mathrm{SD} = \begin{array}{|c|c|c|l}\hline 1 & & & \cdots \\ \hline & \multicolumn{3}{l}{} \\ \cline{1-1} & \multicolumn{3}{l}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{3}{l}{} \end{array}$,

then its steady-state time is either $t = 0$ or $t = 1$.

Example

Such permutations include noncrossing involutions and column reading words of standard tableaux.

Both $v = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$v = 21435 = (12)(34)$ and $w = 31425$ is the column reading word of $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$.

Maximum steady-state time

Theorem (UConn 2020)

If $n \geq 5$ and

$$\mathrm{Q}(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1}\end{array},$$

then the steady-state time of w is $n-3$.

Conjecture

For $n \geq 4$, the steady-state time of a permutation in S_n is at most $n-3$.

Box-Ball System Example ($t = 0$ through 5)

Let $w = 452361$. Then $\mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ and the steady-state time of w is $3 = n - 3$.

$t = 0$	4	5	2	3	6	1															
$t = 1$			4	5		2	1	3	6												
$t = 2$					4	5	2			1	3	6									
$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

Questions (soliton decomposition)

- When is the soliton decomposition SD a standard tableau?
- Can we classify permutations with standard SD using pattern avoidance?
- Classify the permutations with the same soliton decompositions

When is SD(w) a standard tableau?

Example

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \mathrm{SD}(21435) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \qquad \mathrm{SD}(31425) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Theorem (UConn 2020 + D. Grinberg)

Given a permutation w, the following are equivalent:

1. $\mathrm{SD}(w)$ is standard
2. $\mathrm{SD}(w) = \mathrm{P}(w)$
3. the shape of $\mathrm{SD}(w)$ is equal to the shape of $\mathrm{P}(w)$

Definition

We say that a permutation w is *BBS-good* (or *"good"* for short) if the tableau $\mathrm{SD}(w)$ is standard.

$Q(w)$ determines whether w is good

Proposition

Given a standard tableau T, either

$$\mathrm{SD}(w) \text{ is standard for all } w \text{ such that } \mathrm{Q}(w) = T,$$

or

$$\mathrm{SD}(w) \text{ is not standard for all } w \text{ such that } \mathrm{Q}(w) = T.$$

Definition (good tableaux)

A standard tableau T is *good* if each permutation whose Q tableau equals T is good.

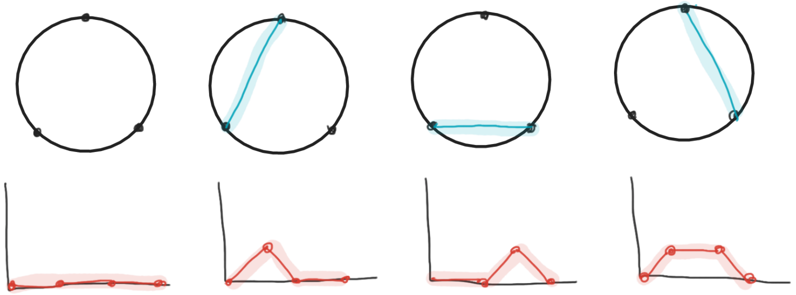
- Question: How many good tableaux are there?

Answer: Good tableaux are counted by the Motzkin numbers!

Work in preparation (SUMRY 2022)

The good standard tableaux, $\{\mathrm{Q}(w) \mid w \in S_n \text{ and } \mathrm{SD}(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

$$M_0 = 1, \qquad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}$$



$$n = 3$$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

Future: Characterize good permutations using pattern avoidance

(Skipped slide)

A pattern v is a *consecutive pattern* of a permutation w if w has a consecutive subsequence whose elements are in the same relative order as v. Otherwise, w *avoids* v.

- $w = 314\mathbf{5926}87$ contains $v = 2413$ because the subsequence 5926 is ordered in the same way as 2413
- $w = 314592687$ avoids $v = 321$ because 314592687 has no consecutive subsequence ordered in the same way as 321.
  (Remark: 314592687 contains a non-consecutive subsequence with pattern 321. What is this subsequence?)

Further question: Come up with a statement “a permutation is good iff it avoids the consecutive patterns ...”

Knuth moves

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- A *Knuth move* between two $v, w \in S_n$ is the act of swapping consecutive entries

$$yxz \text{ and } yzx \qquad \text{(Knuth move of \textit{the first kind}) or}$$
$$xzy \text{ and } zxy \qquad \text{(Knuth move of \textit{the second kind})}$$

where $x < y < z$, or

$$y_1xzy_2 \text{ and } y_1zxy_2 \qquad \text{(Knuth move of \textit{both kinds} } (K_B))$$

where $x < y_1, y_2 < z$.

- We say v and w are *Knuth equivalent* if they differ by a sequence of Knuth moves.

Example

$326\mathbf{514} \sim^{K_2} 326\mathbf{154}$ $\qquad$ $\mathbf{326}154 \sim^{K_1} \mathbf{362}154$ $\qquad$ $362\mathbf{154} \sim^{K_B} 362\mathbf{514}$

P-tableaux and Knuth moves

(skipped slide)

Theorem (Knuth, 1970)

- *There is a path of Knuth moves from w to the row reading word of $P(w)$.*
- *Two permutations have the same P tableau iff they are in the same Knuth equivalence class.*

Example

The Knuth equivalence class of the row reading word $r = 362514$ of $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}$:

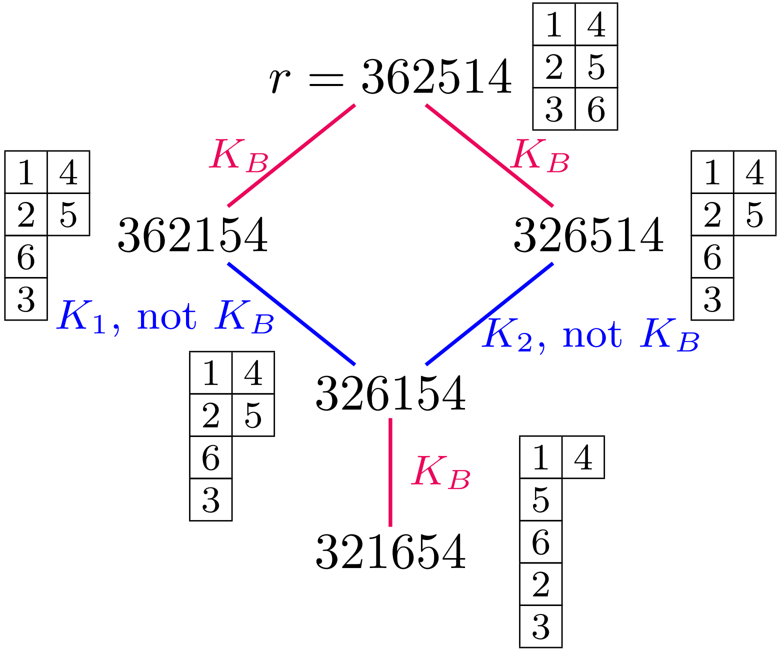
$$\begin{tikzcd}
 & r = 362514 \arrow[dl, dash, "K_B"'] \arrow[dr, dash, "K_B"] & \\
362154 \arrow[dr, dash, "{K_1,\ \text{not } K_B}"'] & & 326514 \arrow[dl, dash, "{K_2,\ \text{not } K_B}"] \\
 & 326154 \arrow[d, dash, "K_B"] & \\
 & 321654 &
\end{tikzcd}$$

Future: Classify permutations with the same soliton decomposition

Partial Result (UConn 2020): The soliton decomposition is preserved by non-K_B Knuth moves, but one K_B move changes the soliton decomposition.

Example

Soliton decompositions of the Knuth equivalence class of 362154:



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Question: Classify permutations with the same soliton decomposition

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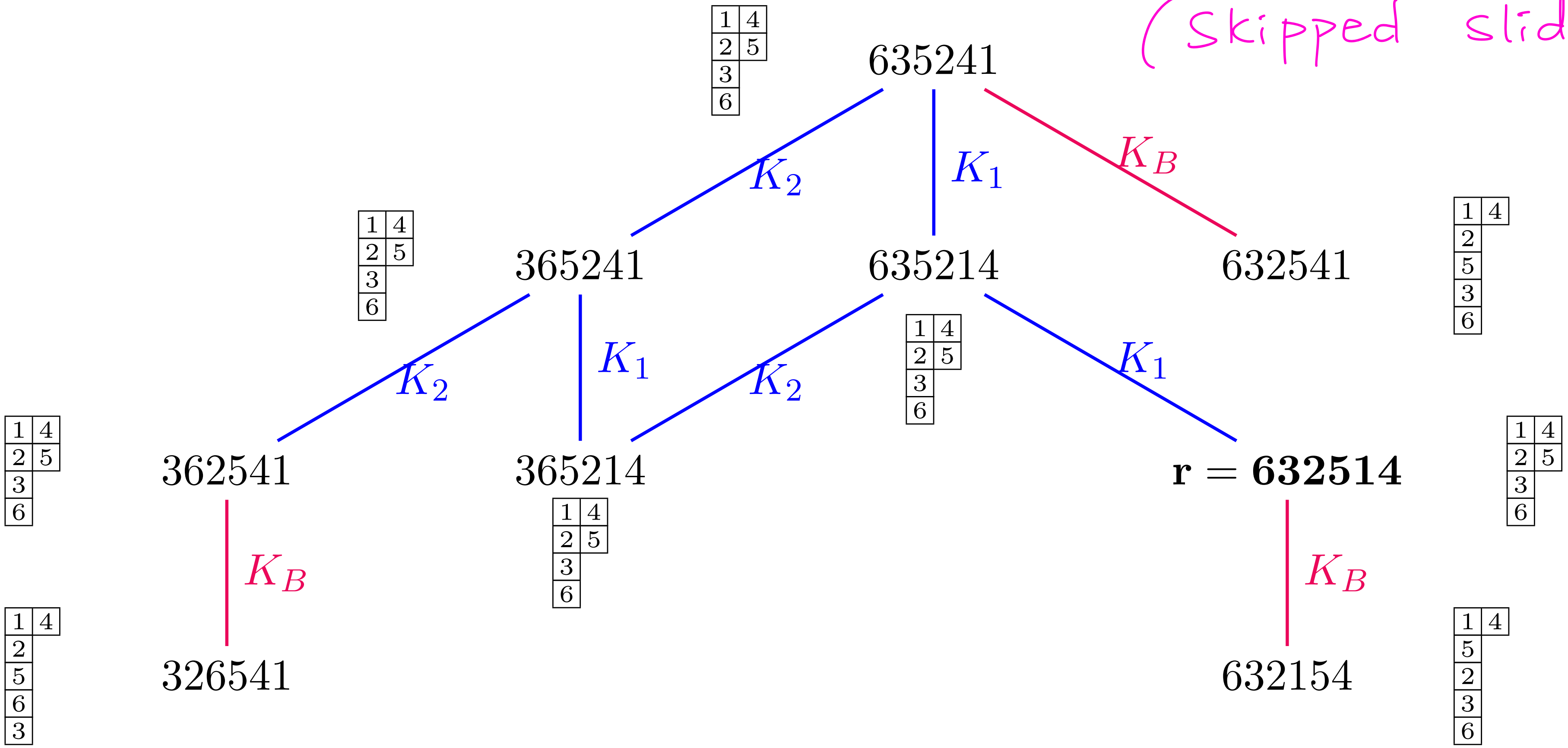

The Knuth equivalence class of $r = 632514$, with their soliton decompositions

The end of part I

Y	O	U	!
A	N	K	
T	H		

Part II: Algebraic combinatorics inspired by Catalan objects

Emily Gunawan, University of Oklahoma,
joint with G. Muller (on cluster algebras, 2020–present),
E. Barnard, E. Meehan, R. Schiffler (on type A quivers and Cambrian posets, 2018–2021),
E. Barnard, R. Coelho Simões, R. Schiffler (on more general quivers, 2021–present)

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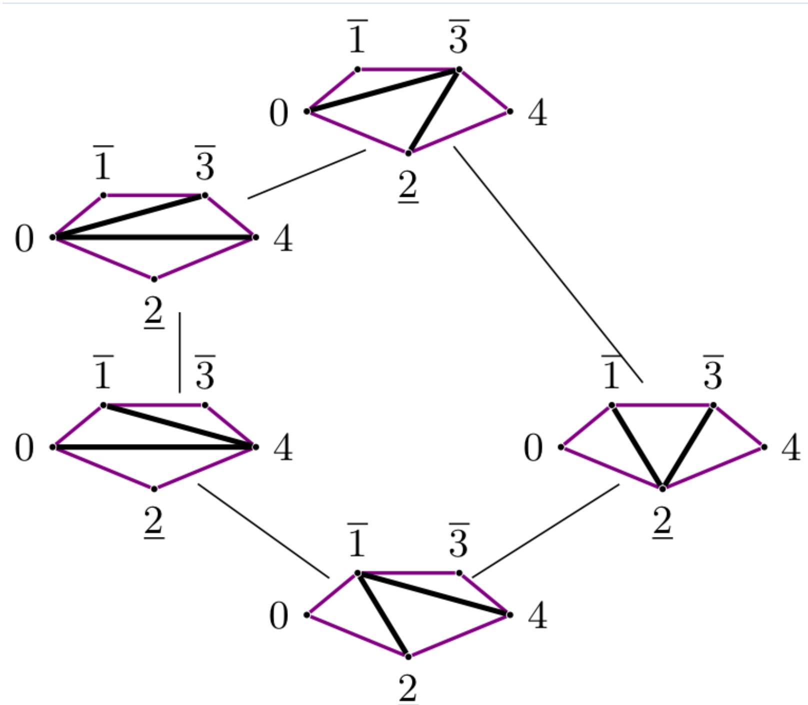
Inspiration: Type A Cambrian poset (Björner—Wachs 1997, Reading 2004)



Def

The *n-th Catalan number* is the number of triangulations of an $(n+2)$-gon

Catalan objects are objects that are counted by the Catalan numbers.

Type A_2 Tamari poset (a special case of Cambrian posets)

Path algebra :

Q quiver e.g. $Q = 1 \xrightarrow{a} 2 \xleftarrow{b} 3 \xleftarrow{c} 4$, $\Bbbk := \mathbb{C}$

<u>Def</u> The path algebra $\Bbbk Q$

- basis : $\{$all paths in Q, including the <u>lazy path</u> e_i at each vertex $i\}$ $= \left\{ \begin{matrix} e_1, & a, & & \\ & e_2, & & \\ & b, & e_3, & \\ & cb, & c, & e_4 \end{matrix} \right\}$
- multiplication on two basis elements : $pp' = \begin{cases} \overset{\text{Concatenation}}{pp'} & \text{if } pp' \text{ is a path} \\ 0 & \text{otherwise} \end{cases}$

$\Bbbk Q \cong$ algebra of matrices of the form $\begin{bmatrix} \lambda_{e_1} & \lambda_a & 0 & 0 \\ 0 & \lambda_{e_2} & 0 & 0 \\ 0 & \lambda_b & \lambda_{e_3} & 0 \\ 0 & \lambda_{cb} & \lambda_c & \lambda_{e_4} \end{bmatrix}$.

each $\lambda_p \in \Bbbk$ is the coefficient of the path p.

Entry in row i, col j $\longleftrightarrow$ path from vertex i to vertex j.

E.g.

$$\begin{array}{cc} & \begin{array}{cccc} & & 3 & \end{array} \\ \begin{array}{c} \\ \\ \\ 4 \end{array} & \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&\lambda_c&0 \end{bmatrix} \\ & \text{path } c \end{array} \begin{array}{cc} & \begin{array}{cccc} & 2 & & \end{array} \\ \begin{array}{c} \\ \\ 3 \\ \end{array} & \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&\lambda_b&0&0 \\ 0&0&0&0 \end{bmatrix} \\ & \text{path } b \end{array} = \begin{array}{cc} & \begin{array}{cccc} & 2 & & \end{array} \\ \begin{array}{c} \\ \\ \\ 4 \end{array} & \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&\lambda_{cb}&0&0 \end{bmatrix} \\ & \text{path } cb \end{array}$$

and

$$\begin{array}{cc} & \begin{array}{cccc} & 2 & & \end{array} \\ \begin{array}{c} \\ \\ 3 \\ \end{array} & \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&\lambda_b&0&0 \\ 0&0&0&0 \end{bmatrix} \\ & \text{path } b \end{array} \begin{array}{cc} & \begin{array}{cccc} & & 3 & \end{array} \\ \begin{array}{c} \\ \\ \\ 4 \end{array} & \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&\lambda_c&0 \end{bmatrix} \\ & \text{path } c \end{array} = \begin{array}{c} \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \\ 0 \end{array}$$

Modules of the path algebra $\Bbbk Q$

A <u>module</u> over an algebra is a generalization of vector space

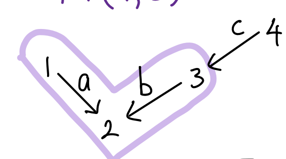
- addition
- Instead of multiplication by a scalar (a number in $\mathbb{R}$ or $\mathbb{C}$), multiply by an element of the algebra (e.g. $\mathbb{Z}$ or $\Bbbk Q$)

In type A ...

"Indecomposable" modules of the path algebra $\Bbbk Q$

$\longleftrightarrow$ intervals $M(i,j)$, $i \le j$ called "strings"

E.g. $M(1,3)$



string ab^{-1} (or $b\bar{a}^{1}$ or ${}^{1}{}_{2}{}^{3}$)

The Auslander–Reiten quiver

The Auslander – Reiten quiver of Q is a directed graph Γ_Q with

vertices: indecomposable modules

arrows: "irreducible morphisms"

$$Q = \begin{tikzcd} v_1 \arrow[dr] & & v_3 \arrow[dl] \arrow[dr] & \\ & v_2 & & v_4 \end{tikzcd}$$

Auslander–Reiten quiver Γ_Q of Q:

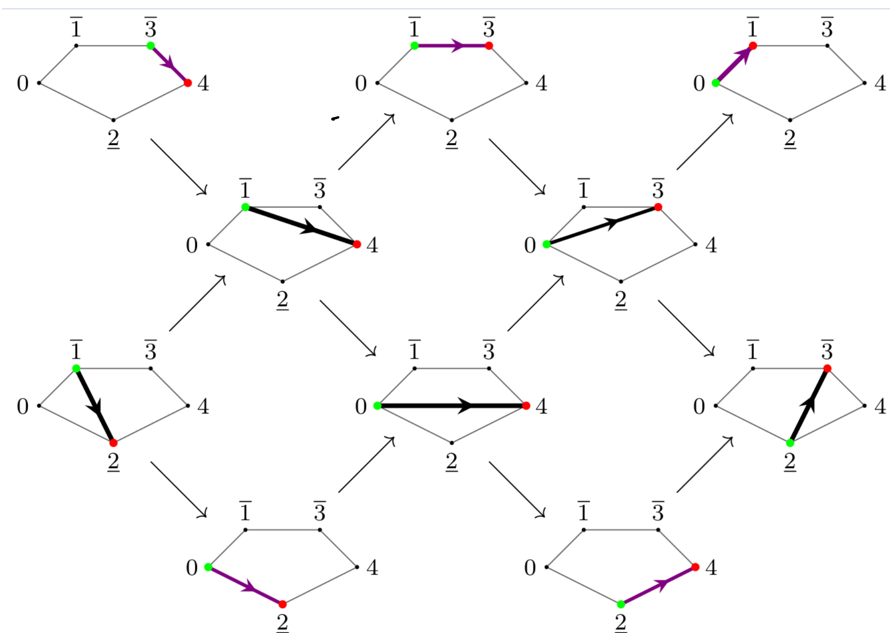
$$\begin{tikzcd} 4 \arrow[dr] & & \substack{3\\2} \arrow[dr] & & 1 \\ & \substack{3\\2\,4} \arrow[ur] \arrow[dr] & & \substack{1\,3\\2} \arrow[ur] \arrow[dr] & \\ 2 \arrow[ur] \arrow[dr] & & \substack{1\;3\\2\;4} \arrow[ur] \arrow[dr] & & 3 \\ & \substack{1\\2} \arrow[ur] & & \substack{3\\4} \arrow[ur] & \end{tikzcd}$$

Barnard — G. — Meehan — Schiffler, 2019 [BGMS 19]

A model for ind Q inspired by the Cambrian posets (for type A)

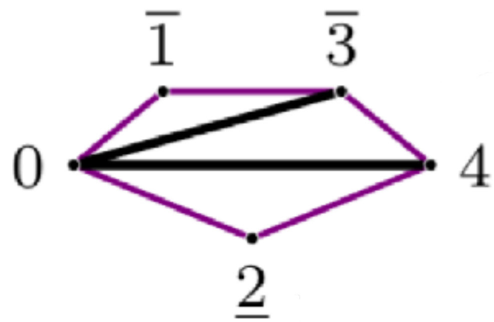
$$\left\{\begin{array}{l}\text{Line segments } \gamma(i,j),\ 0 \leq i < j \leq n+1 \\ \text{(including boundary segments)}\end{array}\right\} \longleftrightarrow \text{ indecomposable modules } M(i+1, j)$$

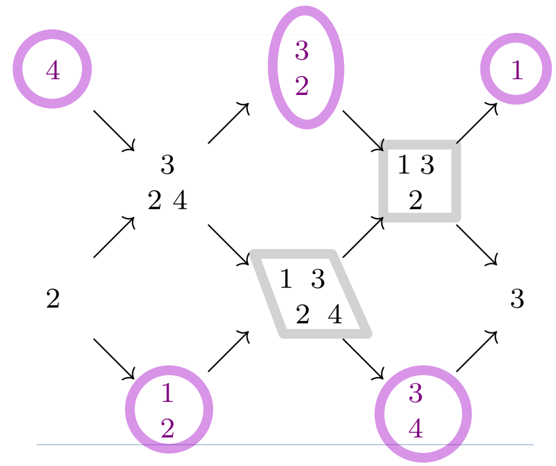
Moving one endpoint counterclockwise $\longleftrightarrow$ irreducible morphisms

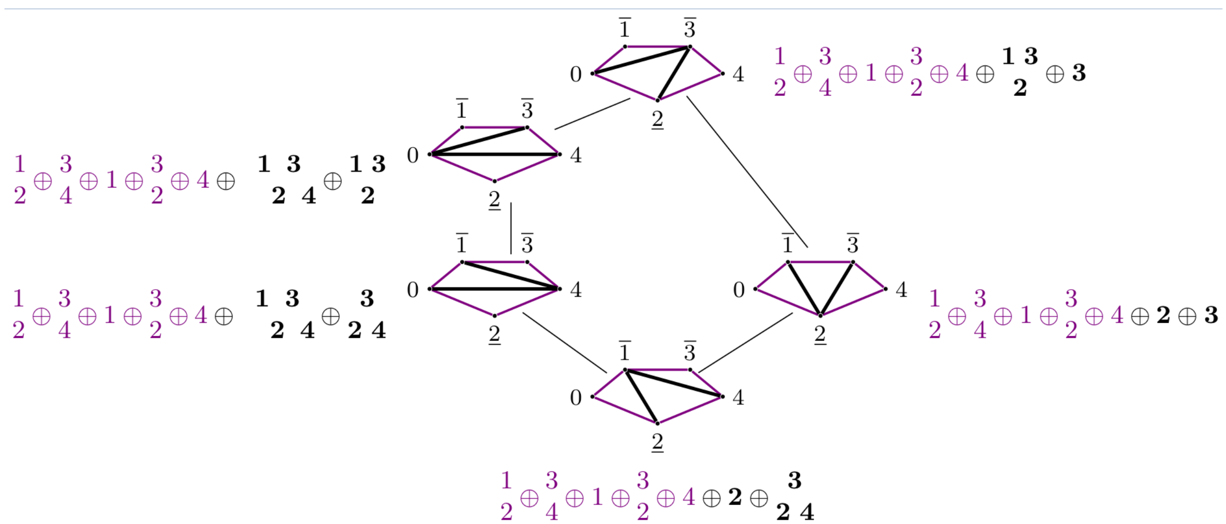


$$\begin{tikzcd}[column sep=small, row sep=small]
4 \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & 1 \\
& \begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1&&3\\&2&&4\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \\
& \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Triangulations $\longleftrightarrow$??







A new class of modules

$\Bbbk Q$: path algebra of type A

<u>Classical def</u> $T \in \mathrm{mod}(\Bbbk Q)$ is <u>maximal rigid</u> if

(T1) T has $|Q_0|$ non-isomorphic summands
\# of vertices of Q

(T2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$

called <u>rigid</u>

<u>Def</u> [BGMS 19]

$T \in \mathrm{mod}(\Bbbk Q)$ is <u>maximal almost rigid</u> (mar) if

(M1) T has $|Q_0| + |Q_1|$ non-isomorphic summands
\# of vertices of Q \# of arrows of Q

(M2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence,
then $E \cong B \oplus A$ or E is indecomposable

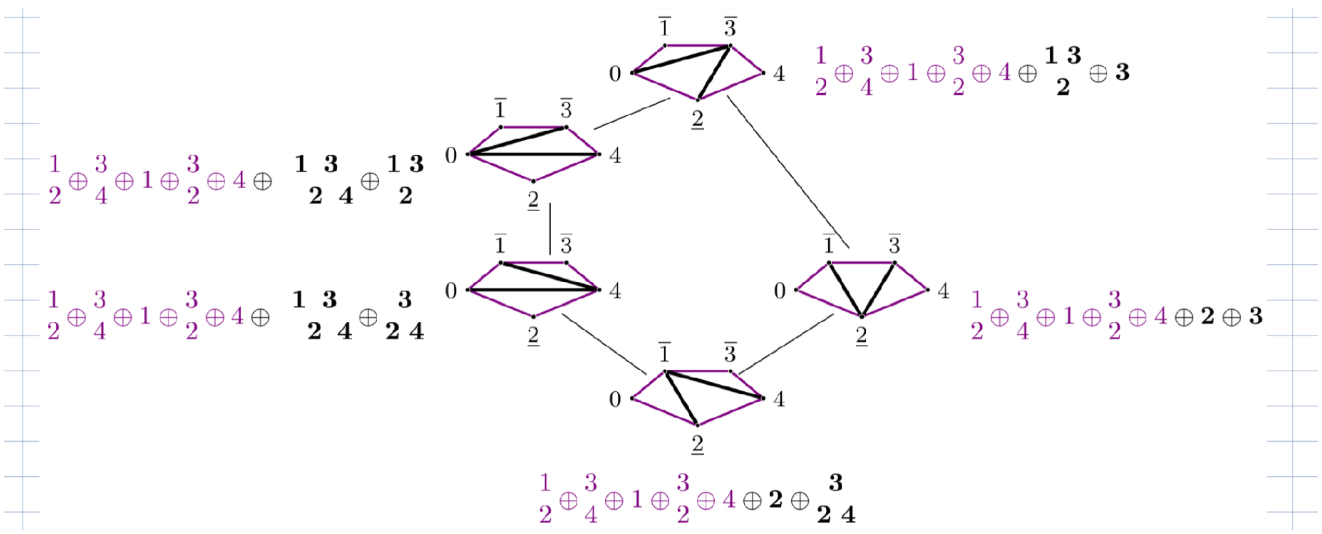
called <u>almost rigid</u>

<u>Rem</u> (M1) can be replaced with :
"T is maximal with respect to (M2)"

Thm [BGMS 19] $\left\{\begin{matrix}\text{triangulations of } P(Q) \\ \text{including boundary edges}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{mar modules} \\ \mathrm{mar}(\Bbbk Q)\end{matrix}\right\}$

Corollary The mar modules (type A) are Catalan objects

There is a natural Cambrian poset structure we can put on the mar modules

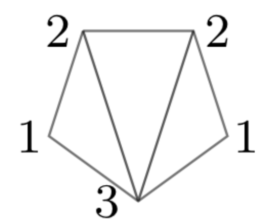


Current & future work [With E. Barnard, R. Coelho Simões, R. Schiffler 2021 – now]

Tell a similar story about mar modules for "gentle algebras", "string algebras", and more.

Conway–Coxeter frieze pattern (1970s)

$$
\begin{array}{ccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots & \\
 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & \textbf{(Row 2)} & & \\
 & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \cdots &
\end{array}
$$



rules : • each diamond $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ satisfies $ad-bc=1$

• only positive integers are allowed

* Conway–Coxeter frieze patterns are Catalan objects!

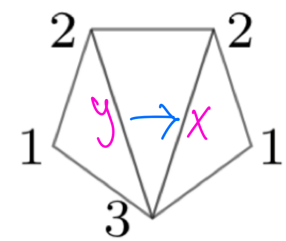
Why?

2nd row of a frieze pattern with n non-trivial rows $\longleftrightarrow$ triangulation of an $(n+3)$-gon

Fomin-Zelevinsky cluster algebras (2001)

* Replace the non-trivial integers with Laurent polynomials:

$$\begin{array}{ccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& \frac{x+y+1}{xy} & & x & & \frac{y+1}{x} & & \frac{x+1}{y} & & y & & \frac{x+y+1}{xy} & \\
\cdots & & \frac{x+1}{y} & & y & & \frac{x+y+1}{xy} & & x & & \frac{y+1}{x} & & \cdots \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$



* The diamond rule still holds, e.g. $\frac{x+1}{y} \cdot y - x \cdot 1 = 1$

* These five Laurent polynomials are called cluster variables.

* "Def"

Dynkin diagram of type A_2 : •—•

The type A_2 cluster algebra is the subring of the field of rational functions $\mathbb{Z}(x,y)$ generated by these five cluster variables

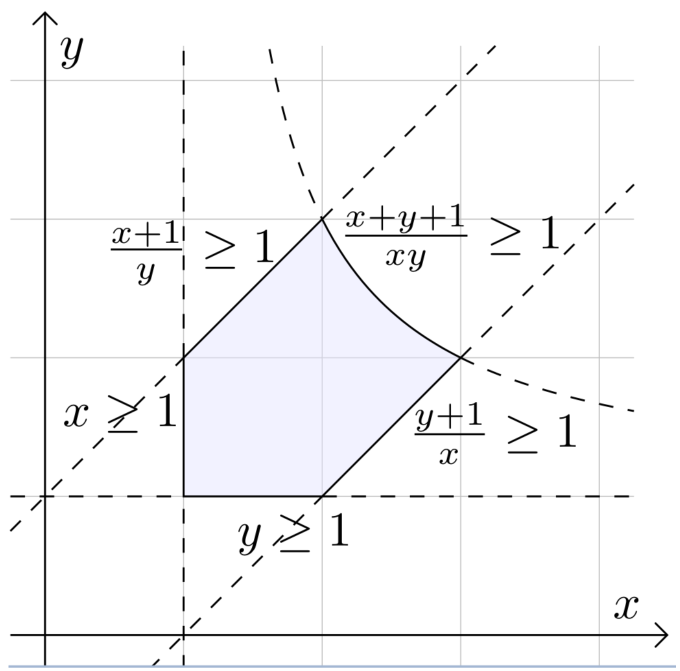
(Skipped slide)

"Def" [G.-Muller 2022]

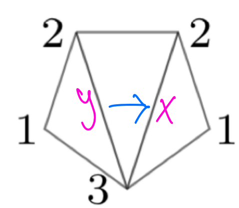
The superunitary region of the A_2 cluster algebra, embedded in $\mathbb{R}^2$

The five cluster variables, each set to $\geqslant 1$:

- x
- y
- $\frac{1+x}{y}$
- $\frac{y+1+x}{xy}$
- $\frac{1+y}{x}$



quiver: $x \to y$ & triangulation:



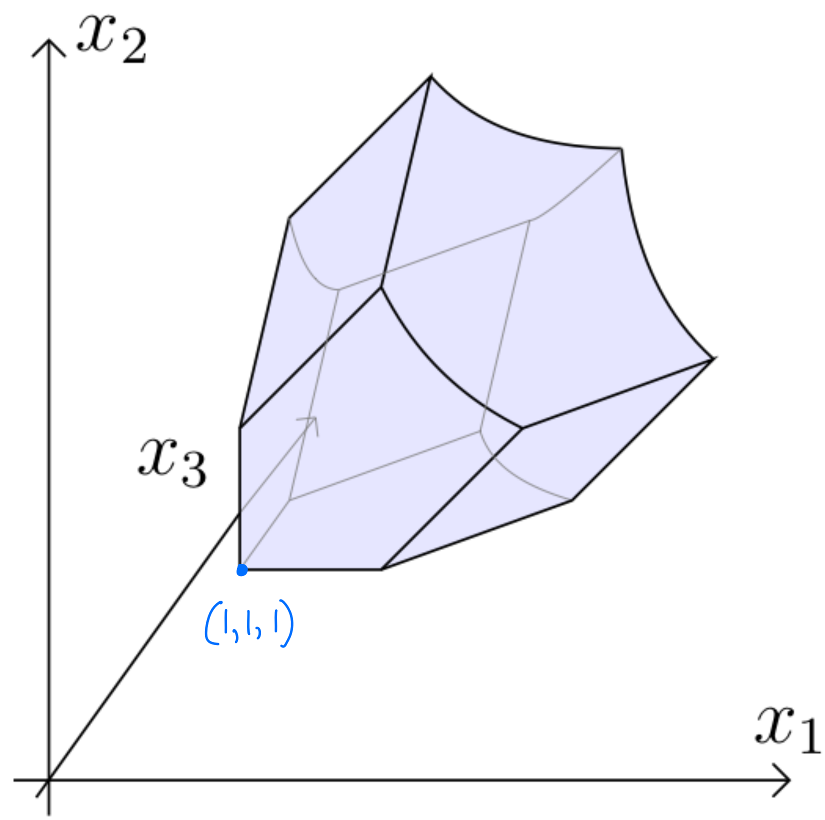
The superunitary region of the A_3 cluster algebra, embedded in $\mathbb{R}^3$

The nine cluster variables, each set to $\geqslant 1$:

$$x_1 \geqslant 1 \qquad x_2 \geqslant 1 \qquad x_3 \geqslant 1$$

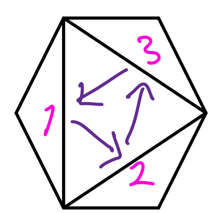
$$\frac{x_2+x_3}{x_1} \geqslant 1 \qquad \frac{x_1+x_3}{x_2} \geqslant 1 \qquad \frac{x_1+x_2}{x_3} \geqslant 1$$

$$\frac{x_1+x_2+x_3}{x_1x_2} \geqslant 1 \qquad \frac{x_1+x_2+x_3}{x_2x_3} \geqslant 1 \qquad \frac{x_1+x_2+x_3}{x_1x_3} \geqslant 1$$



quiver:
$$\begin{tikzcd} & X_3 \arrow[dl] & \\ X_1 \arrow[rr] & & X_2 \arrow[ul] \end{tikzcd}$$

& triangulation:

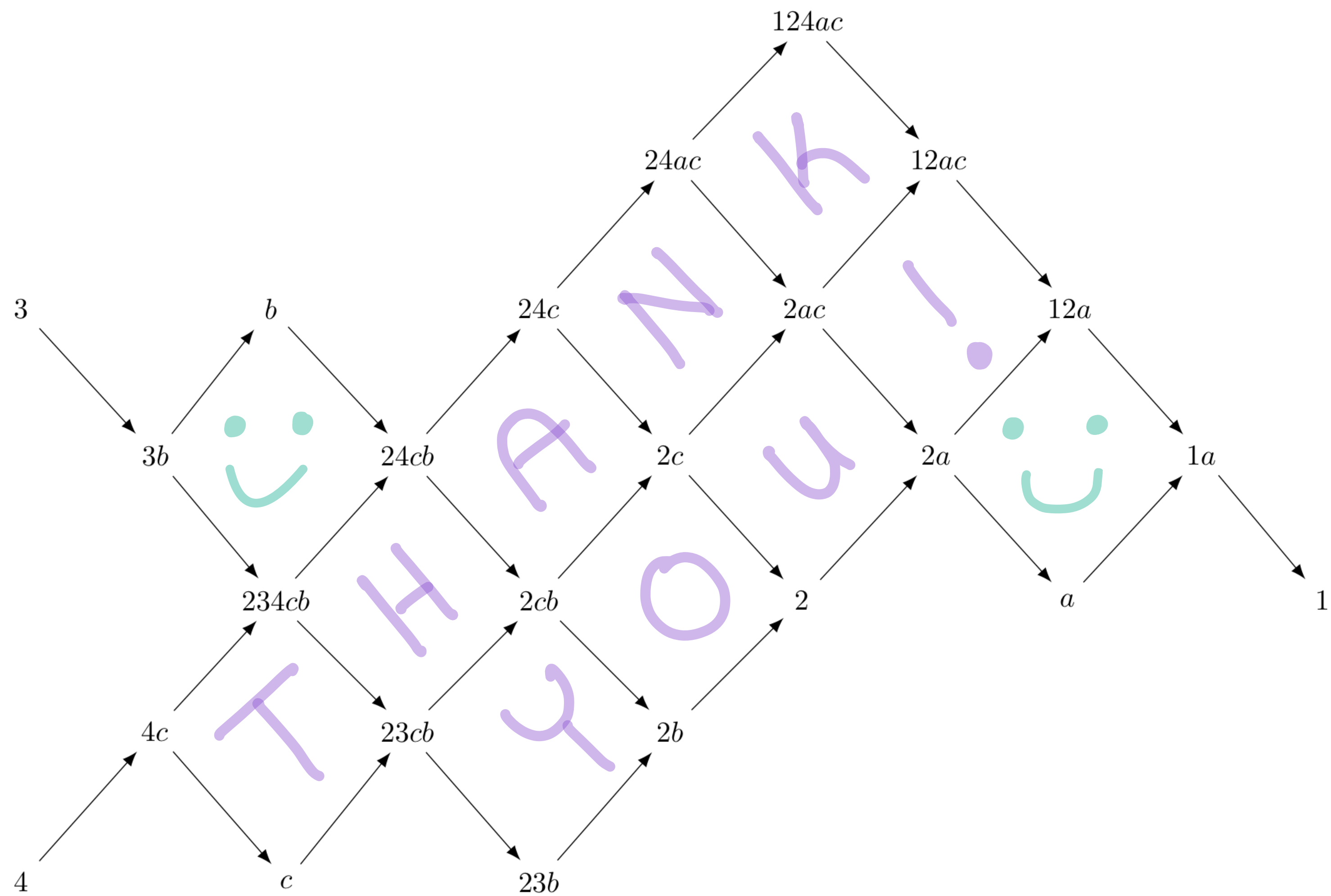


Note: Δ Dynkin diagram ABCDEFG $\longleftrightarrow$ [Fomin-Zelevinsky 2001] cluster algebra $\mathcal{A}$ of finite type Δ

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\text{ [G.-Muller 2022]}\\ \text{The superunitary region of a finite type cluster algebra is}\\ \text{a topological polytope with the same face structure}\\ \text{as the associahedron}\end{array}}$$

<u>Future work:</u>

* Prove that the superunitary region is contained in the convex hull of its extreme points
* Study the superunitary regions (no longer bounded) of infinite type cluster algebras
* We used this theorem to give a uniform proof of a conjecture that there are finitely many positive integral friezes of type ABCDEFG. Can we apply it to other questions?


124ac
24ac
12ac
3
b
24c
2ac
12a
3b
24cb
2c
2a
1a
234cb
2cb
2
a
1
4c
23cb
2b
4
c
23b
THANK
YOU!