From permutations to waves, triangulations, and representations

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Clemson University Mathematical and Statistical Sciences Colloquium
Friday, February 24, 2023

Permutations

Let S_n denote the set of permutations on the numbers $\{1, \ldots, n\}$.

We will represent permutations in two ways,

- in *two-line notation*, as
$$\begin{pmatrix} 1 & 2 & \ldots & n \\ w(1) & w(2) & \ldots & w(n) \end{pmatrix}, \text{ and}$$
- in *one-line notation*, as $w = w(1)w(2) \cdots w(n) \in S_n$.

Example

A permutation in S_5

- in two-line notation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, and
- in one-line notation: 21435

Part I: Box-ball systems and tableaux

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Solitary waves (solitons)

Scott Russell's first encounter (August 1834)

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped.

[The mass of water in the channel] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, ... and after a chase of one or two miles I lost it in the windings of the channel."

Soliton on the Scott Russell Aqueduct on the Union Canal (July 1995)

(ma.hw.ac.uk/solitons/press.html)

Two soliton animation: www.desmos.com/calculator/86loplpajr

(Multicolor) box-ball system, Takahashi 1993

A *box-ball system* is a dynamical system of box-ball configurations.

- At each configuration, balls are labeled by numbers 1 through n in an infinite strip of boxes.
- Each box can fit at most one ball.

Example

A possible box-ball configuration:

$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**									$\cdots$

Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.

$t = 0$	$\cdots$	4	5	2	3	6	1				$\cdots$
	$\cdots$	4	5	2	3	6		1			$\cdots$
	$\cdots$	4	5		3	6	2	1			$\cdots$
	$\cdots$	4	5			6	2	1	3		$\cdots$
	$\cdots$		5	4		6	2	1	3		$\cdots$
	$\cdots$			4	5	6	2	1	3		$\cdots$
$t = 1$	$\cdots$			4	5		2	1	3	6	$\cdots$

Box-ball moves ($t = 0$ through $t = 5$)

$t = 0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**																$\cdots$
$t = 1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**													$\cdots$
$t = 2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**										$\cdots$
$t = 3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**							$\cdots$
$t = 4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**				$\cdots$
$t = 5$	$\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future box-ball moves.

Example

The strings **4**, **25**, and **136** are solitons:

$t = 3$ $\cdots$							**4**	**2**	**5**				**1**	**3**	**6**							$\cdots$
$t = 4$ $\cdots$								**4**		**2**	**5**					**1**	**3**	**6**				$\cdots$
$t = 5$ $\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

After a finite number of box-ball moves, the system reaches a *steady state* where:

- each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

Question (steady-state time)

The time when a permutation w first reaches steady state is called the *steady-state time* of w.

- Find a formula to compute the steady-state time of a permutation, without needing to run box-ball moves.

Tableaux (English notation)

Definition

- A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Nonstandard Tableau:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 6 & 7 \\ \hline 4 \\ \cline{1-1} \end{array}$$

Soliton decomposition

Definition

To construct *soliton decomposition* $\mathrm{SD}(w)$ of w, start with the one-line notation of w, and run box-ball moves until we reach a steady state; the 1st row of $\mathrm{SD}(w)$ is the rightmost soliton, the 2nd row of $\mathrm{SD}(w)$ is the next rightmost soliton, and so on.

Example

$t=0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**													$\cdots$
$t=1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**										$\cdots$
$t=2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**							$\cdots$
$t=3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**				$\cdots$
$t=4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**		$\cdots$

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \text{ with shape } (3,2,1).$$

RSK bijection

The classical Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $w =$ **452361**. Then

$$\mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \multicolumn{1}{|c}{} \\ \cline{1-2} 4 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} \quad \text{and} \quad \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 6 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} .$$

RSK bijection example

Let $w = \mathbf{452361}$.

$$
\mathrm{P}: \quad \mathbf{4} \qquad \begin{array}{cc} 4 & \mathbf{5} \end{array} \qquad \begin{array}{cc} \mathbf{2} & 5 \\ 4 & \end{array} \qquad \begin{array}{cc} 2 & \mathbf{3} \\ 4 & 5 \end{array} \qquad \begin{array}{ccc} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{array} \qquad \begin{array}{ccc} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \qquad \mathrm{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}
$$

$$
\mathrm{Q}: \quad \mathbf{1} \qquad \begin{array}{cc} 1 & \mathbf{2} \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \mathbf{3} & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & \mathbf{4} \end{array} \qquad \begin{array}{ccc} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{array} \qquad \mathrm{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}
$$

Insertion and bumping rule for P

- Insert x into the first row of P.
- If x is larger than every element in the first row, add x to the end of the first row.
- If not, replace the smallest number larger than x in row 1 with x. Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

Q(w) determines the box-ball dynamics of w

Theorem (SUMRY 2021)

If $\mathrm{Q}(v) = \mathrm{Q}(w)$, then

- v and w first reach steady state at the same time, and
- the soliton decompositions of v and w have the same shape

Example

$$v = 21435 \text{ and } w = 31425$$

$$\mathrm{Q}(v) = \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

Both v and w have steady-state time $t = 1$

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

Questions (steady-state time)

Two permutations are said to be *Q-equivalent* if they have the same Q-tableau.

- Given a Q-tableau, find a formula to compute the steady-state time for all permutations in this Q-tableau equivalence class.
- Find an upper bound for steady-state times of all permutations in S_n.

L-shaped soliton decompositions

Theorem (SUMRY 2021)

If a permutation has an L-shaped soliton decomposition $\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\cdots$ $\vdots$,

then its steady-state time is either $t = 0$ or $t = 1$.

Remark

Such permutations include "noncrossing involutions" and "column words" of standard tableaux.

Example

Both $v = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$v = (12)(34)$ and $w = 31425$ is the column word of $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$.

Maximum steady-state time

Theorem (UConn 2020)

If $n \geq 5$ and

$$\mathrm{Q}(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1} \end{array},$$

then the steady-state time of w is $n-3$.

Conjecture

For $n \geq 4$, the steady-state time of a permutation in S_n is at most $n-3$.

A permutation with steady-state time $n-3$

Let $w = 452361 \in S_6$. Then $\mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ and the steady-state time of w is $3 = n-3$.

$t=0$	$\cdots$	**4**	**5**	**2**	**3**	**6**	**1**																$\cdots$
$t=1$	$\cdots$			**4**	**5**		**2**	**1**	**3**	**6**													$\cdots$
$t=2$	$\cdots$					**4**	**5**	**2**			**1**	**3**	**6**										$\cdots$
$t=3$	$\cdots$							**4**	**2**	**5**				**1**	**3**	**6**							$\cdots$
$t=4$	$\cdots$								**4**		**2**	**5**					**1**	**3**	**6**				$\cdots$
$t=5$	$\cdots$									**4**			**2**	**5**						**1**	**3**	**6**	$\cdots$

Questions (soliton decomposition)

- When is the soliton decomposition SD a standard tableau?
- Characterize the permutations with the same soliton decompositions

When is SD(w) a standard tableau?

Example

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \mathrm{SD}(21435) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \mathrm{SD}(31425) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Theorem (UConn 2020 + D. Grinberg)

Given a permutation w, the following are equivalent:

1. $\mathrm{SD}(w)$ is standard
2. $\mathrm{SD}(w) = \mathrm{P}(w)$
3. the shape of $\mathrm{SD}(w)$ is equal to the shape of $\mathrm{P}(w)$

Definition (good permutations)

We say that a permutation w is *good* if the tableau $\mathrm{SD}(w)$ is standard.

Q(w) determines whether w is good

Proposition

Given a standard tableau T, either

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are good,}$$

or

$$\text{All } w \text{ such that } \mathrm{Q}(w) = T \text{ are not good.}$$

Definition (good tableaux)

A standard tableau T is *good* if $T = \mathrm{Q}(w)$ and w is good.

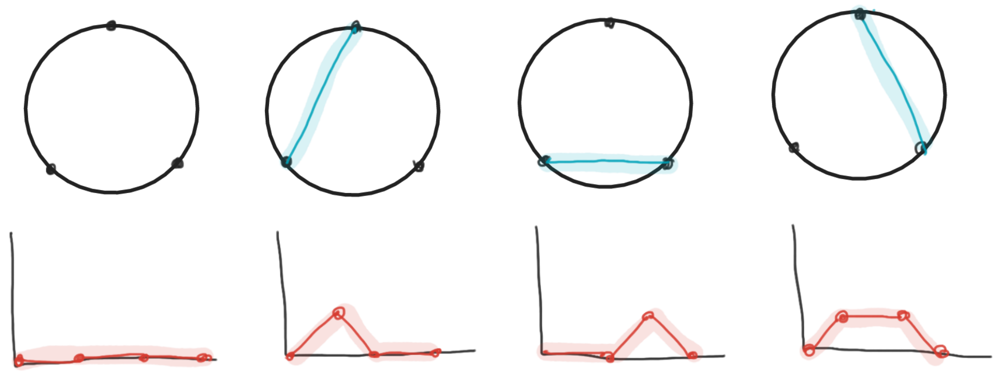
- Question: How many good tableaux are there?

Answer: Good tableaux are new Motzkin objects!

Theorem (SUMRY 2022)

The good standard tableaux, $\{\mathrm{Q}(w) \mid w \in S_n \text{ and } \mathrm{SD}(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

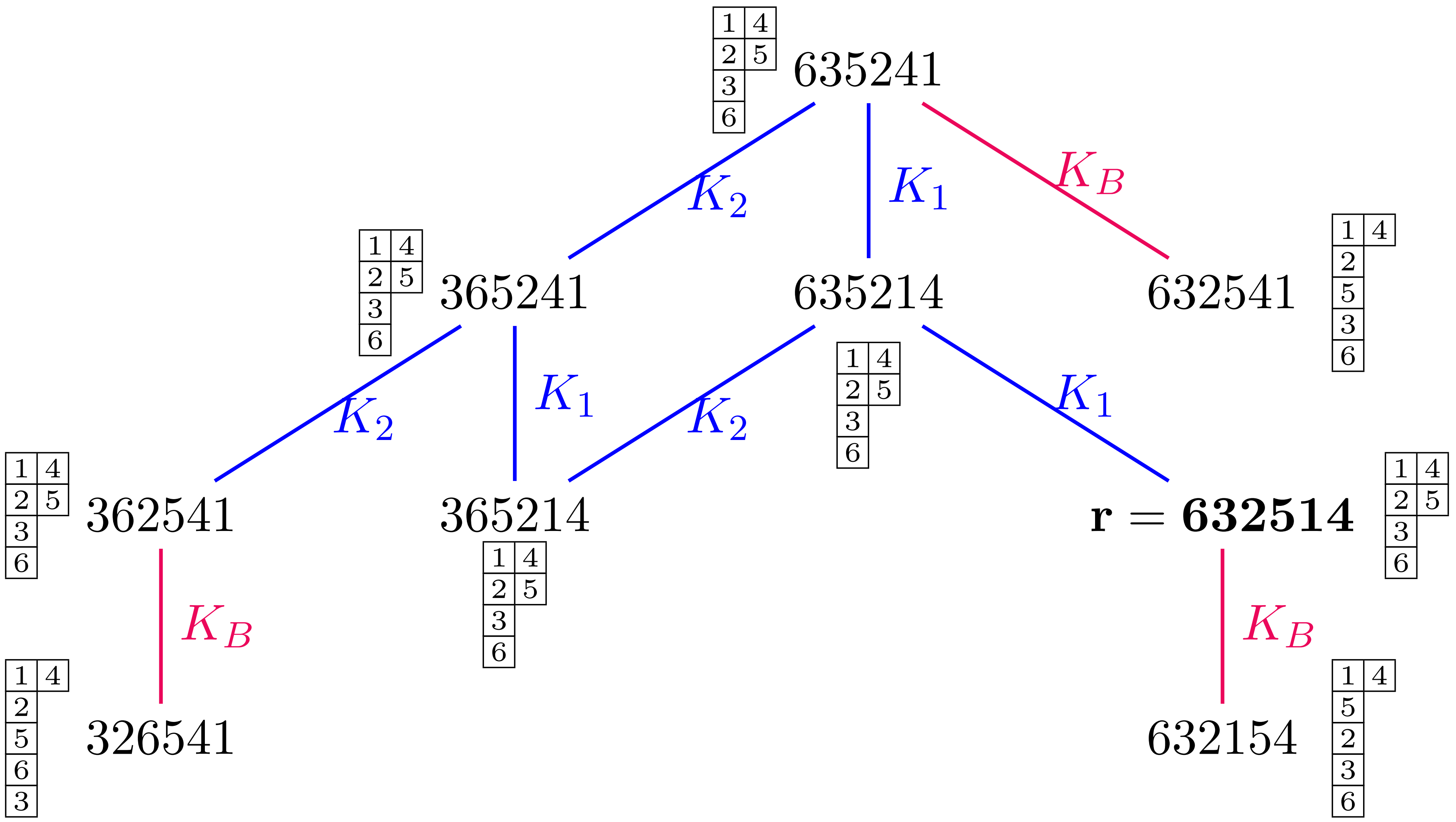
$$M_0 = 1, \qquad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}$$



$$M_3 = 4$$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

Further question: Characterize permutations with the same soliton decomposition



Permutations connected by *Knuth moves* to $\mathbf{r} = \mathbf{632514}$ and their soliton decompositions

The end of part I

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A	N	K	
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Knuth moves

- A *Knuth move* between two $v, w \in S_n$ is the act of swapping consecutive entries

$$yxz \text{ and } yzx \qquad \text{(Knuth move of } \textit{the first kind}\text{) or}$$

$$xzy \text{ and } zxy \qquad \text{(Knuth move of } \textit{the second kind}\text{)}$$

  where $x < y < z$, or

$$y_1xzy_2 \text{ and } y_1zxy_2 \qquad \text{(Knuth move of } \textit{both kinds } (K_B)\text{)}$$

  where $x < y_1, y_2 < z$.

- We say v and w are *Knuth equivalent* if they differ by a sequence of Knuth moves.

Example

$326\mathbf{51}4 \sim^{K_2} 326\mathbf{15}4 \quad \mathbf{326}154 \sim^{K_1} \mathbf{362}154 \quad 362\mathbf{15}4 \sim^{K_B} 362\mathbf{51}4$

P-tableaux and Knuth moves

Theorem (Knuth, 1970)

- *There is a path of Knuth moves from w to the row reading word of $P(w)$.*
- *Two permutations have the same P tableau iff they are in the same Knuth equivalence class.*

Example

The Knuth equivalence class of the row reading word $r = 362514$ of $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}$:

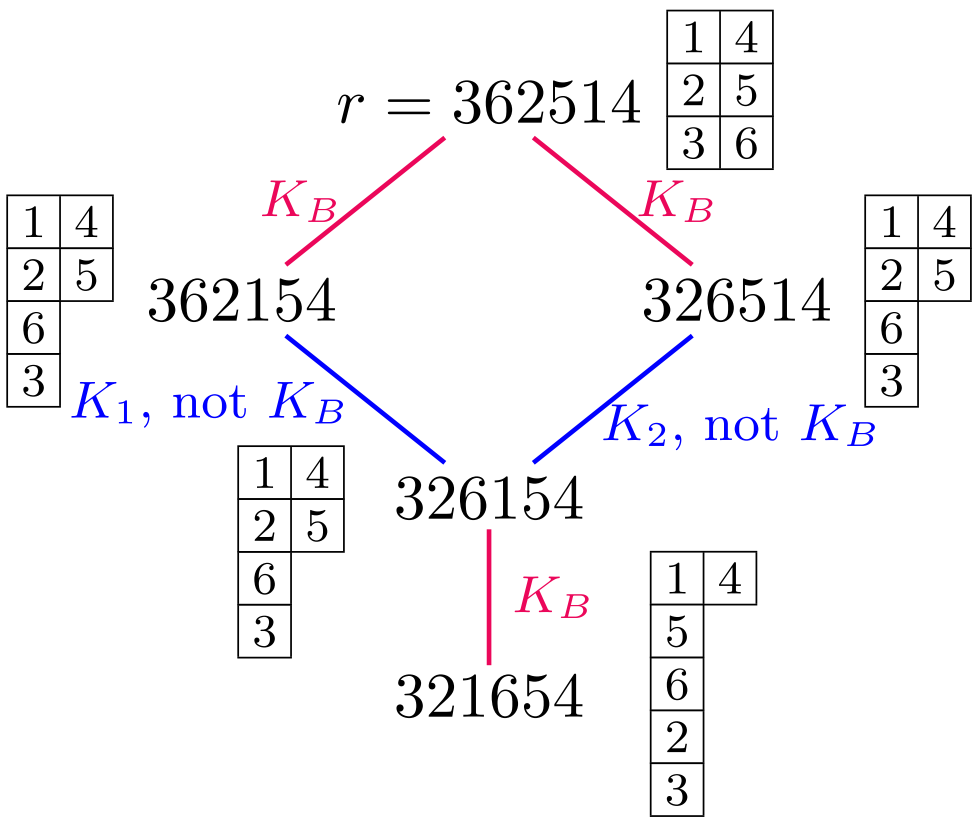
$$\begin{tikzcd}
 & r = 362514 \arrow[dl, dash, "K_B"'] \arrow[dr, dash, "K_B"] & \\
362154 \arrow[dr, dash, "{K_1, \text{ not } K_B}"'] & & 326514 \arrow[dl, dash, "{K_2, \text{ not } K_B}"] \\
 & 326154 \arrow[d, dash, "K_B"] & \\
 & 321654 &
\end{tikzcd}$$

Future: Characterize permutations with the same soliton decomposition

Partial Result (UConn 2020): The soliton decomposition is preserved by non-K_B Knuth moves, but one K_B move changes the soliton decomposition.

Example

Soliton decompositions of the Knuth equivalence class of 362154:



Examples: permutations with L-shaped SD

A permutation with L-shaped SD which is not a column reading word:

$w = 3217654 = (13)(47)(56)$ is a noncrossing involution.

$$\mathrm{P}(w) = \mathrm{Q}(w) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \text{and} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 7 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

An involution which is neither noncrossing nor a column reading word:

$v = 5274163 = (15)(37)$ has a crossing.

$$\mathrm{P}(v) = \mathrm{Q}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \text{and} \qquad \mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 7 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Permutations connected by K_B moves having the same SD

Two permutations with the same SD which are connected by K_B moves:

$r = 35124$ $\qquad \mathrm{SD}(r) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

K_B

$\mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\qquad 31524$

K_2, not K_1

$\mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\qquad 31254$

K_2, not K_1

$\mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\qquad 13254$

K_B

$w = 13524$ $\qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

Future: Characterize good permutations using pattern avoidance

A pattern v is a *consecutive pattern* of a permutation w if w has a consecutive subsequence whose elements are in the same relative order as v. Otherwise, w *avoids* v.

- $w = 314\mathbf{5926}87$ contains $v = 2413$ because the subsequence 5926 is ordered in the same way as 2413
- $w = 314592687$ avoids $v = 321$ because 314592687 has no consecutive subsequence ordered in the same way as 321. (Remark: 314592687 contains a non-consecutive subsequence with pattern 321. What is this subsequence?)

Further question: Come up with a statement "a permutation is good iff it avoids the consecutive patterns ..."

Part II: Triangulations and quiver representations

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Clemson University Mathematical and Statistical Sciences
Colloquium
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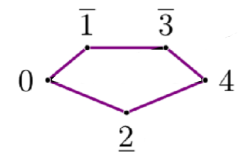
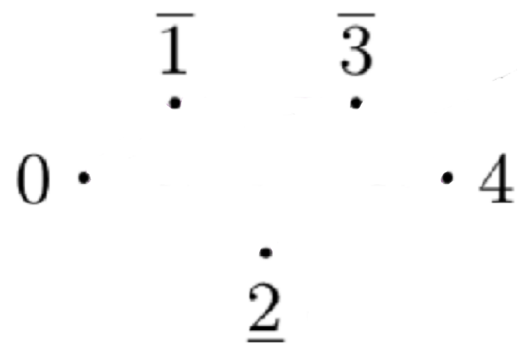
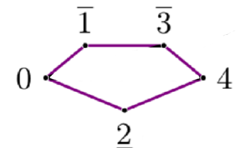
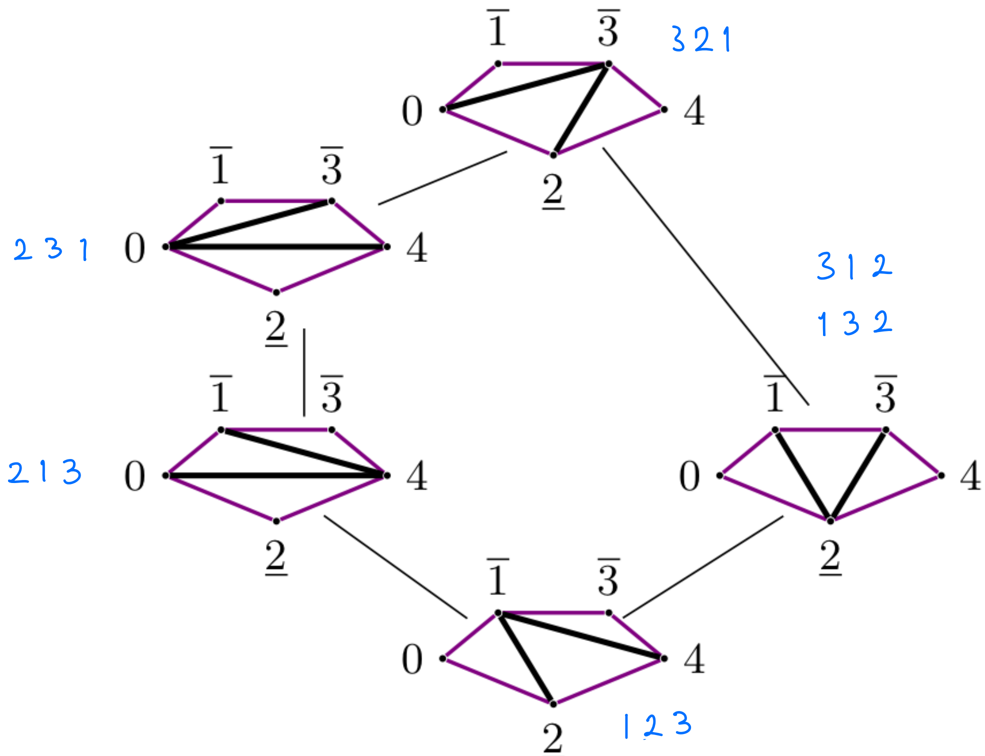
Inspiration: Type A η map (Björner – Wachs 1997, Reading 2004)

Surjection $\eta : S_3 \longrightarrow\!\!\!\!\rightarrow \{$ triangulations of  $\}$



Inspiration: Type A η map (Björner – Wachs 1997, Reading 2004)

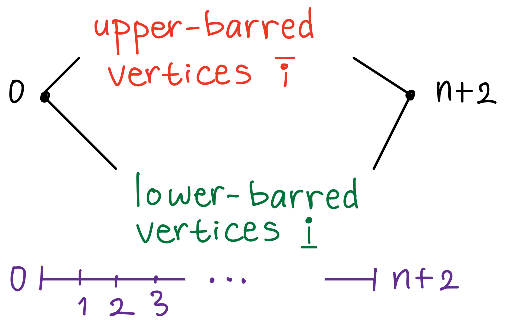
Surjection $\eta : S_3 \longrightarrow\!\!\!\!\rightarrow \{$ triangulations of  $\}$



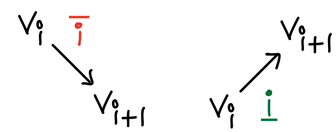
Inspiration: Type A η map (Björner – Wachs 1997, Reading 2004)

* Given a $\underbrace{\text{quiver}}_{\text{directed graph}}$ Q which is an orientation of $\underbrace{v_1 - v_2 - \cdots - v_{n+2}}_{\text{the type } A_{n+2} \text{ Dynkin diagram}}$,

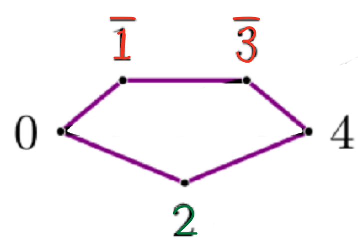
we construct polygon $P(Q)$ with vertices $0, 1, 2, \ldots, n+2$ from left to right



via the rule



<u>Ex</u> $P(Q) =$



$$Q = v_1 \xrightarrow{\ \bar{1}\ } v_2 \xleftarrow{\ \underline{2}\ } v_3 \xrightarrow{\ \bar{3}\ } v_4$$

* In general, we have a surjection $\eta^Q : S_{n+1} \twoheadrightarrow \{\text{triangulations of } P(Q)\}$

Quiver representations

Q quiver <u>ex</u> $Q = \begin{tikzcd}[column sep=small] v_1 \arrow[dr] & & v_3 \arrow[dl] \arrow[dr] & \\ & v_2 & & v_4 \end{tikzcd}$ $\mathbb{k} := \mathbb{C}$

A <u>representation</u> M of Q is assigning

* a finite-dimensional $\mathbb{k}$-vector space to each vertex of Q
* a $\mathbb{k}$-linear map for each arrow of Q

$$M = \begin{tikzcd}[column sep=small] 0 \arrow[dr, "0"] & & \mathbb{k} \arrow[dl, "1"'] \arrow[dr, "1"] & \\ & \mathbb{k} & & \mathbb{k} \end{tikzcd}$$

Fact

If Q is an orientation of the type A

Dynkin diagram $\underset{1}{\bullet}$ — $\underset{2}{\bullet}$ — $\underset{3}{\bullet}$ — $\cdots$ — $\underset{n+2}{\bullet}$,

"Indecomposable" representations of Q

$\longleftrightarrow$

intervals $\{i, i+1, \dots, j\}$, $1 \leq i \leq j \leq n+2$

$$\begin{tikzcd}[column sep=small] \underset{0}{v_1} \arrow[dr, "0"] & & \underset{\mathbb{k}}{v_3} \arrow[dl, "1"'] \arrow[dr, "1"] & \\ & \underset{\mathbb{k}}{v_2} & & \underset{\mathbb{k}}{v_4} \end{tikzcd}$$

<u>Ex</u> $M(2,4) = {}_2{}^3{}_4$

The Auslander–Reiten quiver

The Auslander–Reiten quiver of Q is a directed graph Γ_Q with

vertices: indecomposable representations

arrows: "irreducible morphisms"

$$Q = \begin{tikzcd}[row sep=small] v_1 \arrow[dr] & & v_3 \arrow[dl] \arrow[dr] & \\ & v_2 & & v_4 \end{tikzcd}$$

Auslander–Reiten quiver Γ_Q of Q:

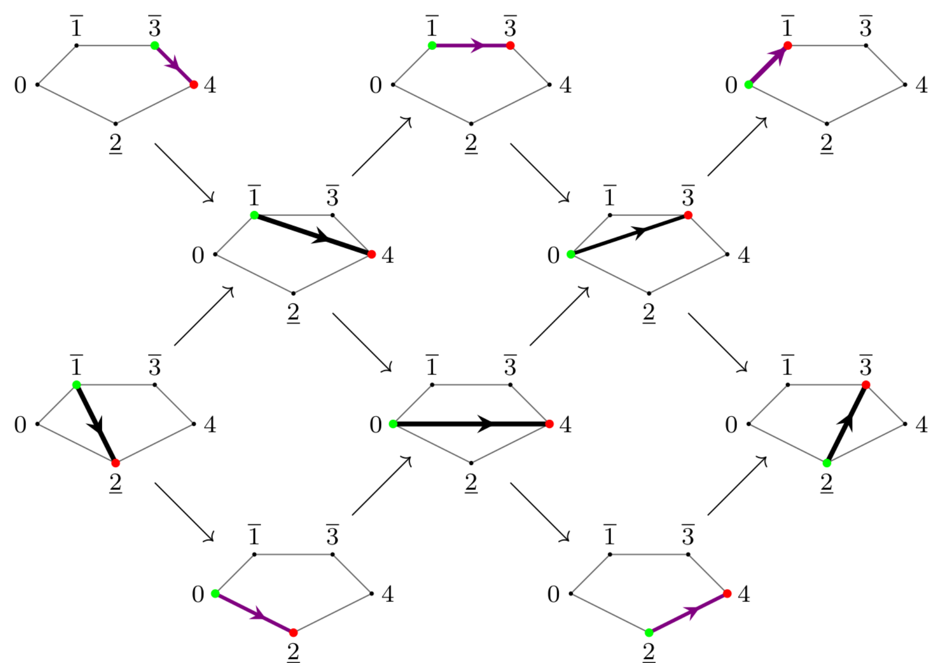
$$\begin{tikzcd}
4 \arrow[dr] & & \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \arrow[dr] & & 1 \\
& \begin{smallmatrix} & 3 & \\ 2 & & 4 \end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] \arrow[dr] & & \begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \\
& \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[ur] & & \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Barnard — G. — Meehan — Schiffler, 2019 [BGMS 19]

A model for the AR quiver inspired by the η map (for type A)

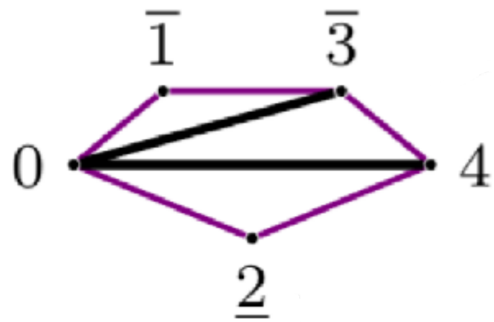
$$\left\{ \text{Line segments } \gamma(i,j) \right\} \longleftrightarrow \left\{ \text{indecomposable representations } M(i+1, j) \right\}$$

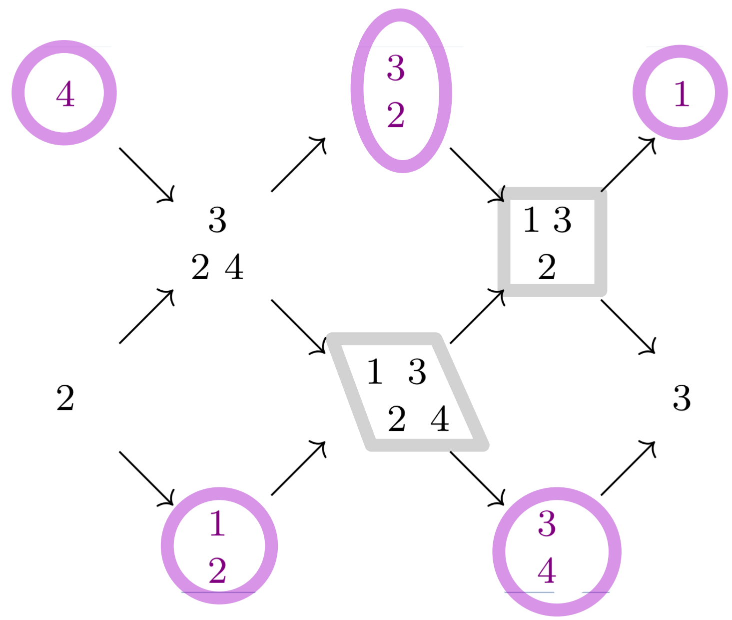
Moving one endpoint counterclockwise $\longleftrightarrow$ irreducible morphisms



$$\begin{tikzcd}
4 \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & 1 \\
& \begin{smallmatrix}3\\2\ 4\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\ \ 3\\2\ \ 4\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \\
& \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Triangulations $\longleftrightarrow$ 





A new class of quiver representations

<u>Def</u> [BGMS 19]

Let Q be a type A quiver.

A representation T of Q is <u>maximal almost rigid</u> (mar) if

(1) T has (# vertices of Q)+(# arrows of Q) non-isomorphic summands

(2) For each pair A, B of summands of T, if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$ or E is indecomposable

called <u>almost rigid</u>

<u>Remark</u> Condition (1) can be replaced with:

"T is maximal with respect to (2)"

Triangulations $\longleftrightarrow$??

Thm [BGMS 19]

$\{$triangulations of $P(Q)\} \longleftrightarrow \{$mar representations of Q (type A)$\}$

Def

The n-th Catalan number is the number of triangulations of the $(n+2)$-gon

Corollary

The mar representations are counted by the Catalan numbers

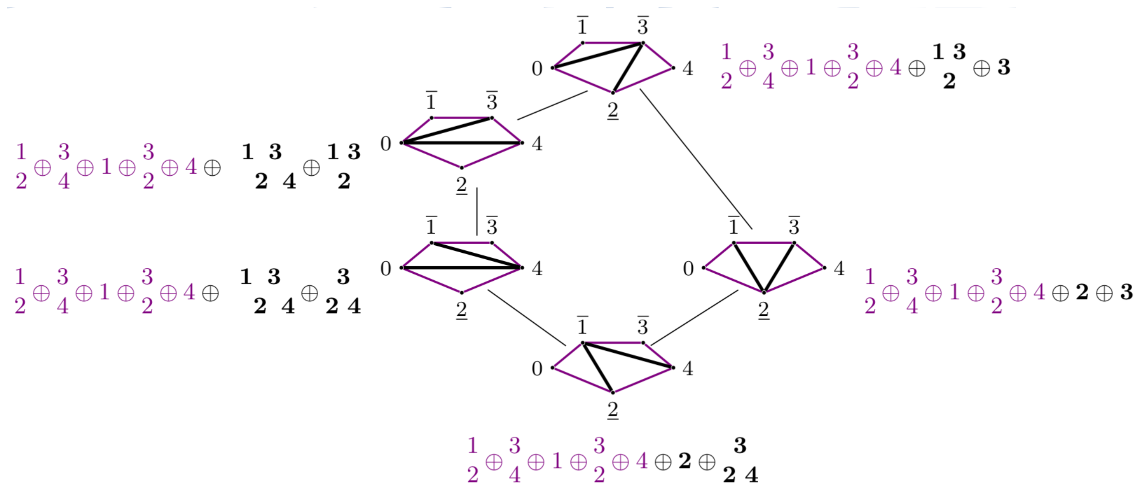
Current & future work With E. Barnard, R. Coelho Simões, R. Schiffler

Tell similar stories about mar objects in the setting of "gentle quivers with relations", "string quivers with relations", and more.

Partial order on the mar representations

Thm [BGMS 19]

We put a natural Cambrian poset structure on the mar representations of Q



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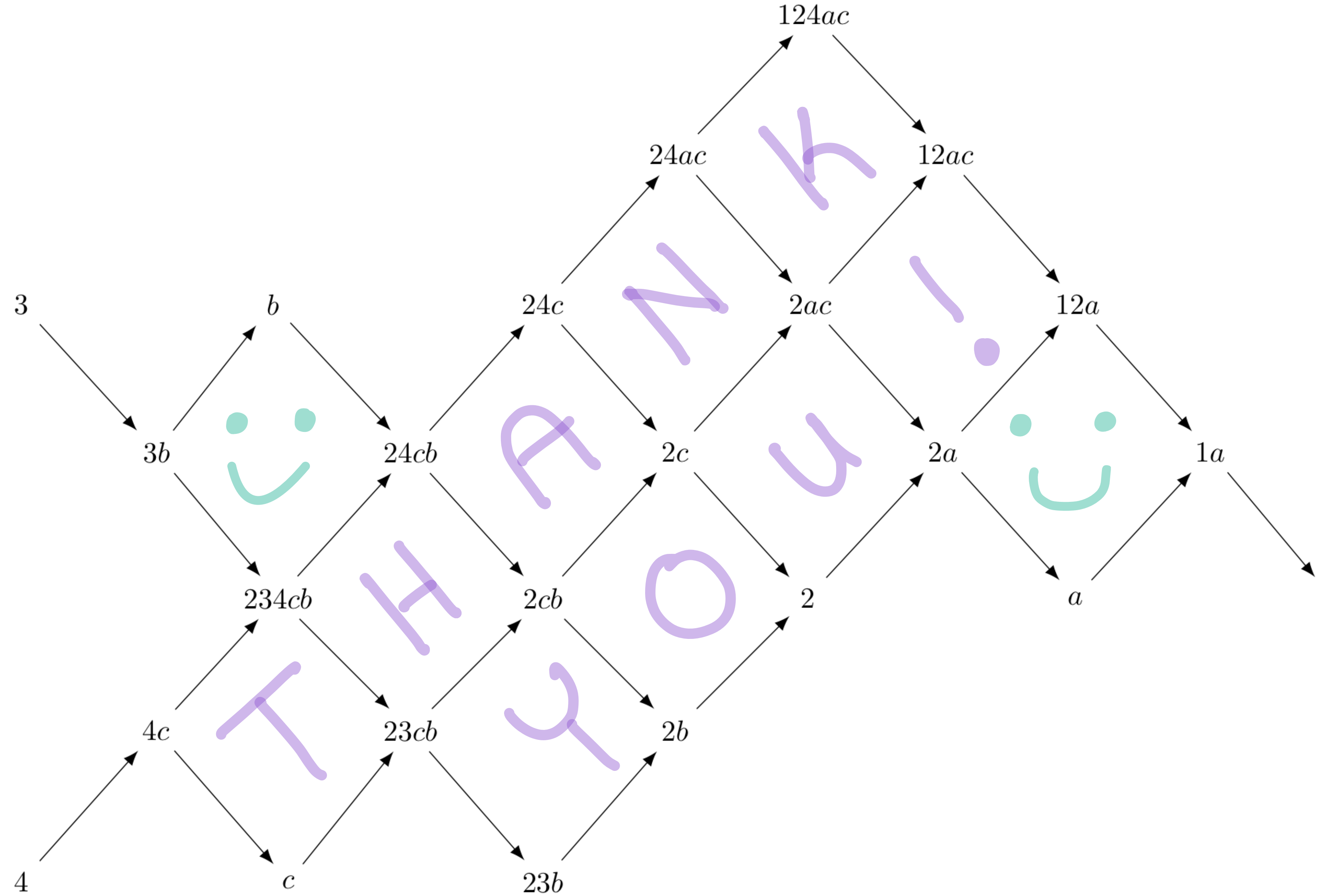
Tell similar stories about mar objects in the setting of "gentle quivers with relations", "string quivers with relations", and more.


124ac
24ac
12ac
3
b
24c
2ac
12a
3b
24cb
2c
2a
1a
234cb
2cb
2
a
1
4c
23cb
2b
4
c
23b
THANK
YOU!