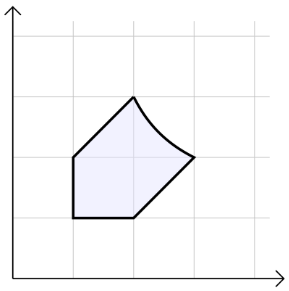

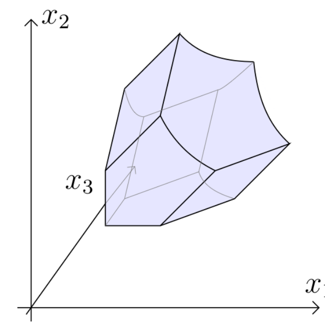

Superunitary regions, generalized associahedra, and friezes of Dynkin type cluster algebras

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Jt. w/ Greg Muller

Special Session on Cluster Algebras, Positivity and Related Topics

AMS Spring Central Sectional Meeting

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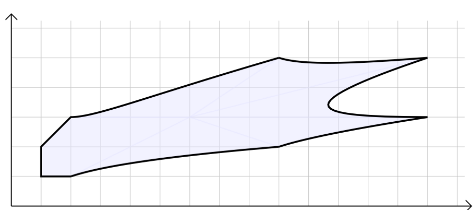
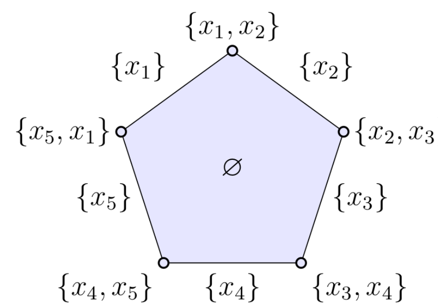




[Fomin-Zelevinsky 2001]

Δ Dynkin diagram $\longleftrightarrow$ cluster algebra $\mathcal{A}$ of finite type Δ

ABCDEFG

E.g. Type C_2 Δ: $a \overset{(2,1)}{\text{———}} b$

Choose an orientation of Δ to get a "valued quiver":

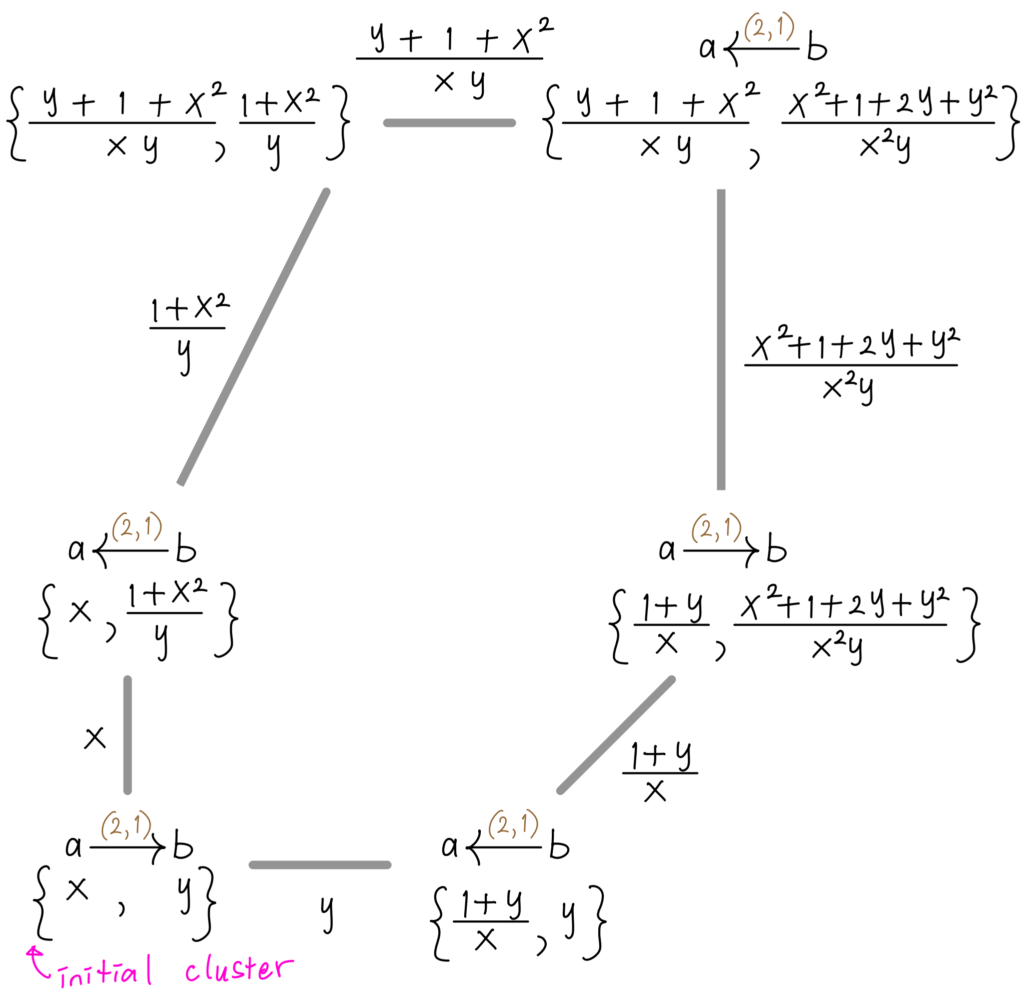
$Q = a \xrightarrow{(2,1)} b$

$\{x, \quad y\}$

initial (valued) quiver

initial cluster

The exchange graph for type C_2 cluster algebra



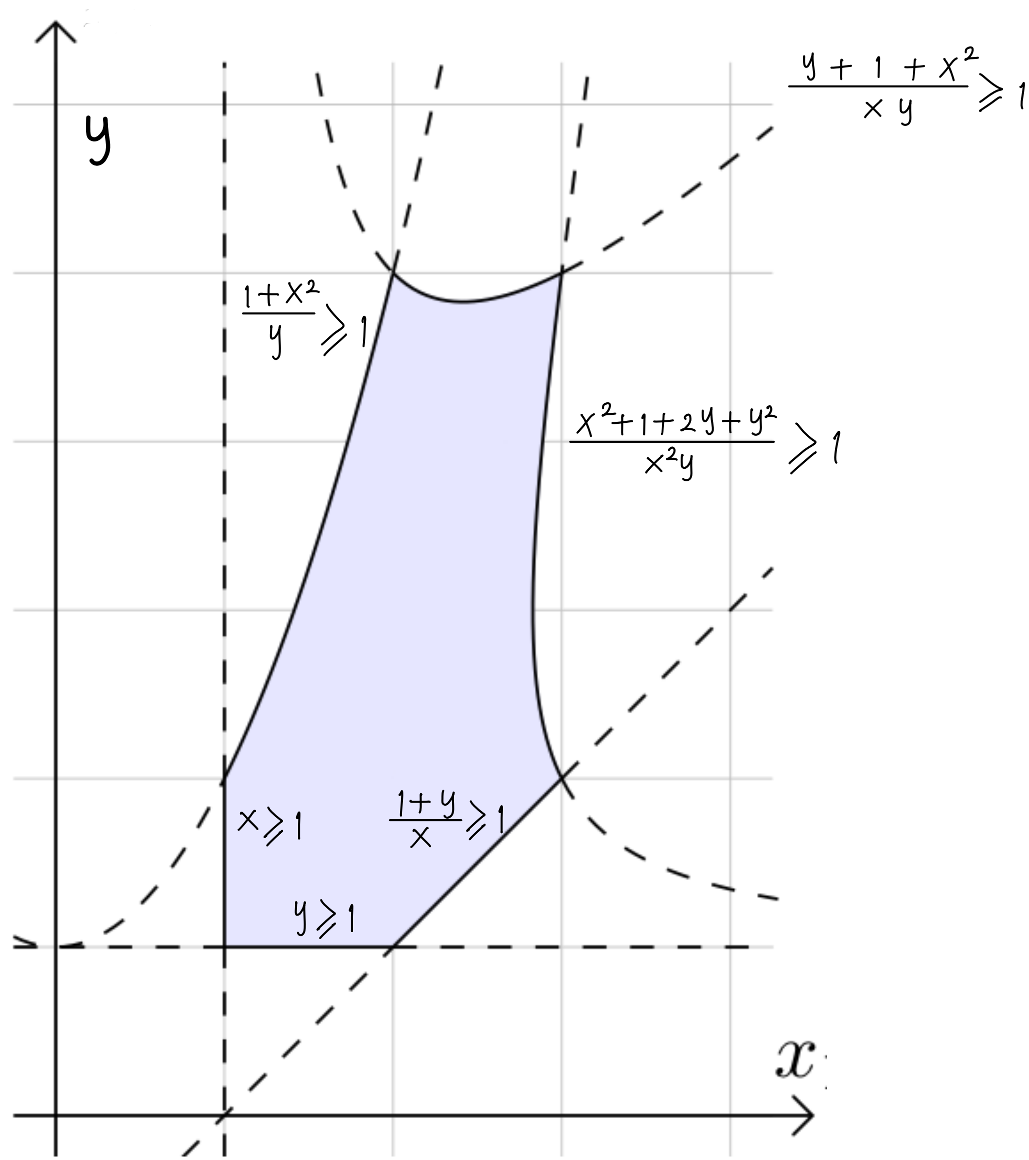
The six cluster variables:

- x
- y
- $\frac{1+x^2}{y}$
- $\frac{y+1+x^2}{xy}$
- $\frac{x^2+1+2y+y^2}{x^2y}$
- $\frac{1+y}{x}$

The superunitary region of the C_2 cluster algebra, embedded in $\mathbb{R}^2$

The six cluster variables, each set to $\geqslant 1$:

- $x \geqslant 1$
- $y \geqslant 1$
- $\dfrac{1+x^2}{y} \geqslant 1$
- $\dfrac{y+1+x^2}{xy} \geqslant 1$
- $\dfrac{x^2+1+2y+y^2}{x^2 y} \geqslant 1$
- $\dfrac{1+y}{x} \geqslant 1$



The superunitary region of the A_3 cluster algebra, embedded in $\mathbb{R}^3$

Initial quiver

$$\begin{tikzcd} & 3 \arrow[dl] & \\ 1 \arrow[rr] & & 2 \arrow[ul] \end{tikzcd}$$

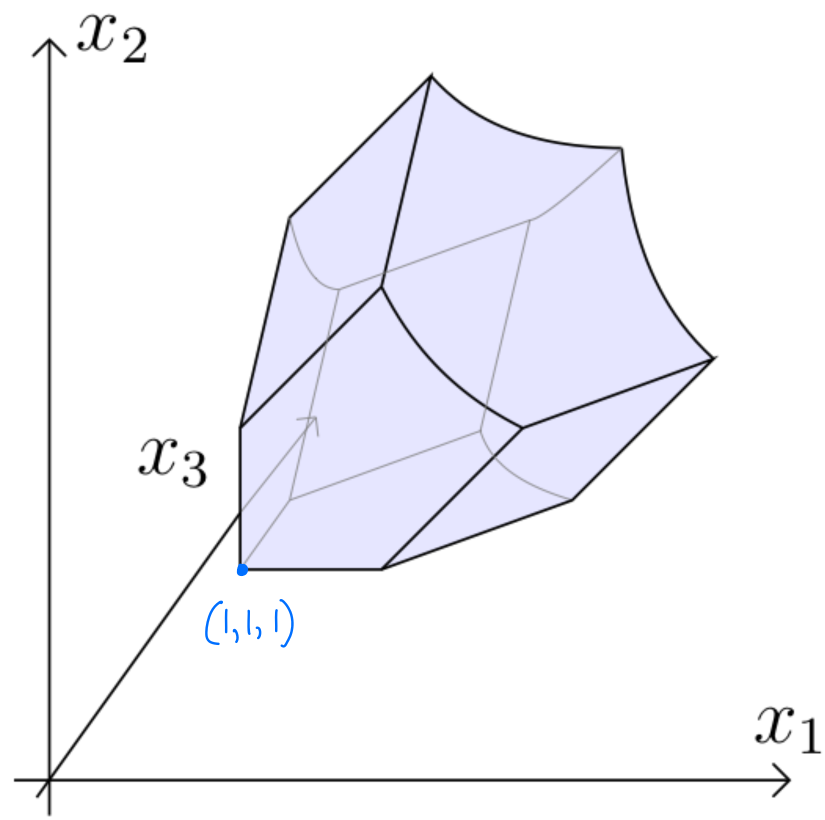
Initial cluster $\{x_1, x_2, x_3\}$

The nine cluster variables, each set to $\geqslant 1$:

$$x_1 \geqslant 1 \qquad x_2 \geqslant 1 \qquad x_3 \geqslant 1$$

$$\frac{x_2+x_3}{x_1} \geqslant 1 \qquad \frac{x_1+x_3}{x_2} \geqslant 1 \qquad \frac{x_1+x_2}{x_3} \geqslant 1$$

$$\frac{x_1+x_2+x_3}{x_1x_2} \geqslant 1 \qquad \frac{x_1+x_2+x_3}{x_2x_3} \geqslant 1 \qquad \frac{x_1+x_2+x_3}{x_1x_3} \geqslant 1$$



[Chapoton — Fomin — Zelevinsky 2002]

Let $\mathcal{A}$ be a Dynkin type cluster algebra.
The exchange graph of $\mathcal{A}$ is the 1-skeleton of a polytope called <u>the generalized associahedron of $\mathcal{A}$.</u>

Faces of the gen. associahedron are indexed by subclusters:

a <u>subcluster</u> is a subset of a cluster

0-face (vertex) $\longleftrightarrow$ cluster $\mathbb{X}=\{x_1,x_2,\dots,x_r\}$

$r :=$ # of vertices of Q

face $\longleftrightarrow$ subcluster

$(r-1)$-face (facet) $\longleftrightarrow$ a cluster variable x

1-face (edge) $\longleftrightarrow$ a subcluster $\mathbb{X}$ of size $r-1$ (aka a mutation)

Interior $\longleftrightarrow$ the empty subcluster

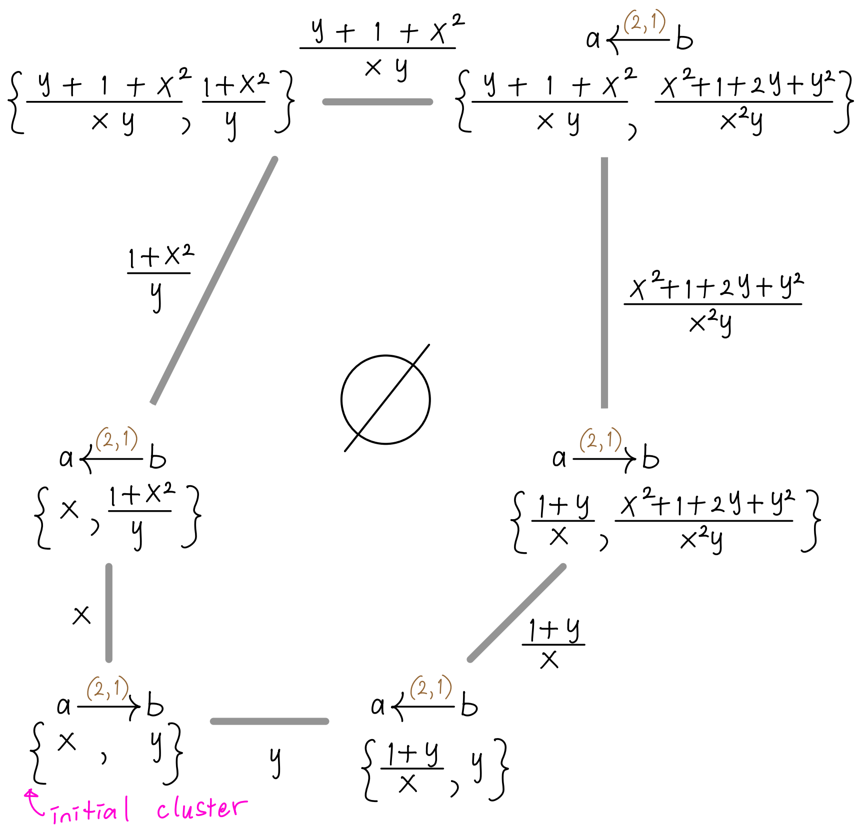
Idea Construct a regular CW complex with the same face structure as the generalized associahedron

E.g. for type C_2 cluster algebra:

vertices $\longleftrightarrow$ clusters

facets $\longleftrightarrow$ cluster variables

interior $\longleftrightarrow$ the empty subcluster



Totally positive region

Def $\mathcal{A}$: a cluster algebra.

- Define a topological space $\mathcal{A}(\mathbb{R}) := \{\text{ring homomorphisms } p : \mathcal{A} \to \mathbb{R}\}$ with the coarsest topology for which, for all $a \in \mathcal{A}$, the map
$$f_a : \mathcal{A}(\mathbb{R}) \longrightarrow \mathbb{R}, \quad p \longmapsto p(a)$$
is continuous.

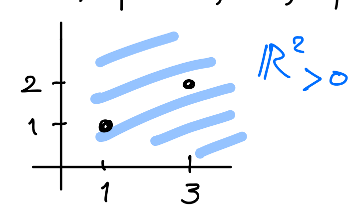
- The <u>totally positive region of $\mathcal{A}$</u> is $\mathcal{A}(\mathbb{R}_{>0}) :=$ the set of ring homomorphisms $p : \mathcal{A} \to \mathbb{R}$ which send each cluster variable to a positive number.

Fact Given a cluster $\mathbf{x} = \{x_1, x_2, \dots, x_r\}$ in $\mathcal{A}$, we can identify $\mathcal{A}(\mathbb{R}_{>0})$ with the positive orthant $\mathbb{R}^r_{>0}$ via homeomorphism
$$f_{\mathbf{x}} : \mathcal{A}(\mathbb{R}_{>0}) \simeq \mathbb{R}^r_{>0}, \quad p \longmapsto (p(x_1), p(x_2), \dots, p(x_r))$$

E.g. $\mathbf{x} = \{x_1, x_2\}$

if $p(x_1) = 1$, $p(x_2) = 1$ then $f_{\mathbf{x}}(p) = (1, 1)$

if $p(x_1) = 3$, $p(x_2) = 2$ then $f_{\mathbf{x}}(p) = (3, 2)$



Superunitary region

<u>Main Def</u> The $\overbrace{\text{superunitary}}^{\text{"bigger than 1"}}$ region of a Dynkin type cluster algebra $\mathcal{A}$ is

$$\underset{\text{superunitary region}}{\mathcal{A}(\mathbb{R}_{\geq 1})} := \left\{ \begin{array}{c} \text{ring homomorphisms } p:\mathcal{A}\to\mathbb{R} \text{ such that} \\ p(x)\geq 1 \text{ for all cluster variables } x \end{array}\right\} \subset \underset{\text{Totally positive region}}{\mathcal{A}(\mathbb{R}_{>0})}$$

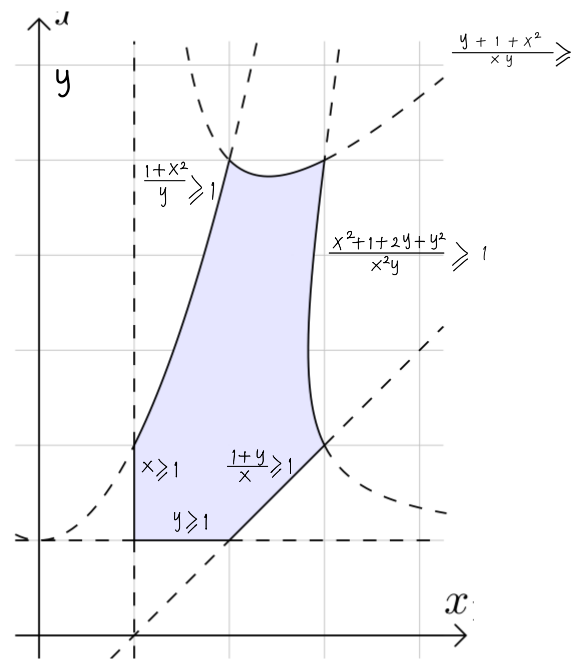
Given a cluster $\mathbf{x}$, use homeomorphism $f_{\mathbf{x}} : \underset{\text{Totally positive region}}{\mathcal{A}(\mathbb{R}_{>0})} \xrightarrow{\sim} \underset{\text{Positive orthant}}{\mathbb{R}^r_{>0}}$

to embed $\underset{\text{superunitary region}}{\mathcal{A}(\mathbb{R}_{\geq 1})}$ into $\mathbb{R}^r_{>0}$:

Set each $\underbrace{\text{cluster variable}}_{\text{(positive Laurent polynomial)}} \geq 1$

E.g. Embedding of type C_2 $\mathcal{A}(\mathbb{R}_{\geq 1})$ into $\mathbb{R}^2$

using $\begin{array}{c} a \xrightarrow{(2,1)} b \\ \mathbf{x} = \{x, y\} \end{array}$:



$\mathbb{X} := \{x, y\}$

In this embedding $f_{\mathbb{X}}$,

p where $\quad p(x) = 3, p(y) = 2 \quad \overset{f_{\mathbb{X}}}{\longmapsto} \quad (3,2) \in \mathbb{R}^2$

(an extreme point)

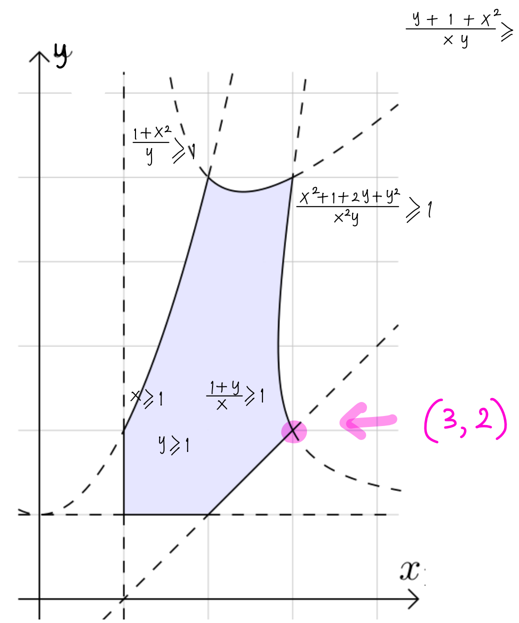
$$p\left(\frac{1+x^2}{y}\right) = \frac{1+9}{2} = 5$$

$$p\left(\frac{y+1+x^2}{xy}\right) = \frac{2+1+9}{6} = 2$$

$$p\left(\frac{x^2+1+2y+y^2}{x^2y}\right) = \frac{9+1+4+4}{18} = 1$$

$$p\left(\frac{1+y}{x}\right) = \frac{1+2}{3} = 1$$

all integer

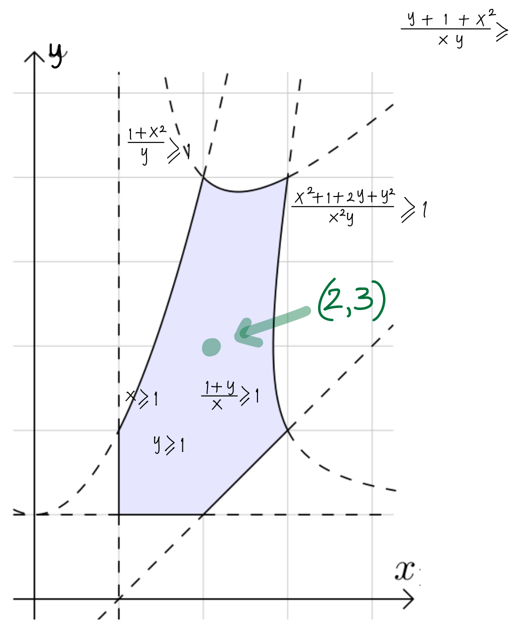


$\mathbb{X} := \{x, y\}$

In this embedding $f_{\mathbb{X}}$,

p where $p(x) = 2, p(y) = 3 \overset{f_{\mathbb{X}}}{\longmapsto} (2,3) \in \mathbb{R}^2$

(an interior point)



$$p\left(\frac{1+x^2}{y}\right) = \frac{1+4}{3} = \frac{5}{3}$$

$$p\left(\frac{y+1+x^2}{xy}\right) = \frac{3+1+4}{6} = \frac{8}{6} = \frac{4}{3}$$

$$p\left(\frac{x^2+1+2y+y^2}{x^2y}\right) = \frac{4+1+6+9}{12} = \frac{20}{12} = \frac{5}{3}$$

$$p\left(\frac{1+y}{x}\right) = \frac{1+3}{2} = 2$$

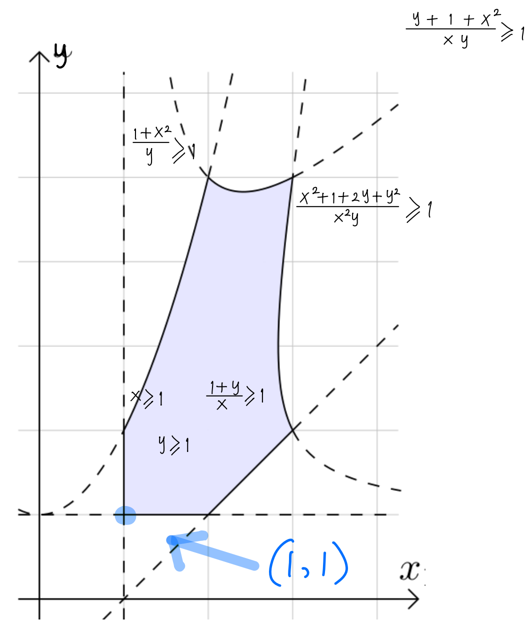
$\mathbb{X} := \{x, y\}$

In this embedding $f_{\mathbb{X}}$,

p where $p(x) = 1,\ p(y) = 1 \xmapsto{f_{\mathbb{X}}} (1, 1) \in \mathbb{R}^2$

(an extreme point)

$$\left.\begin{aligned}
&p\left(\frac{1+x^2}{y}\right) = 2 \\
&p\left(\frac{y+1+x^2}{x\,y}\right) = 3 \\
&p\left(\frac{x^2+1+2y+y^2}{x^2 y}\right) = 5 \\
&p\left(\frac{1+y}{x}\right) = 2
\end{aligned}\right\} \text{ all integer}$$



Thm A If $\mathcal{A}$ is a Dynkin type cluster algebra, the superunitary region $\mathcal{A}(\mathbb{R}_{\geq 1})$ is a regular CW complex which is cellular homeomorphic to the generalized associahedron.

Subcluster face indexed by subcluster $\mathbb{X}$ is $\{p : \mathcal{A} \to \mathbb{R} \text{ s.t } p(a) = 1 \text{ iff } a \in \mathbb{X}\}$

- k-face $\longleftrightarrow$ a subcluster $\mathbb{X}$ of size $r-k$
- $(r-1)$-face (facet) $\longleftrightarrow$ a cluster variable x
- 1-face (edge) $\longleftrightarrow$ a subcluster $\mathbb{X}$ of size $r-1$ (aka a mutation)
- 0-face (vertex) $\longleftrightarrow$ a cluster $\mathbb{X} = \{x_1, x_2, \dots, x_r\}$

- Boundary of $\mathcal{A}(\mathbb{R}_{\geq 1}) = \displaystyle\bigsqcup_{\mathbb{X} \text{ nonempty}}$ subcluster face indexed by $\mathbb{X}$
- Interior of $\mathcal{A}(\mathbb{R}_{\geq 1}) = \{p : \mathcal{A} \to \mathbb{R} \text{ where } p(x) > 1 \ \forall \text{ cl. var } x\}$ indexed by the empty subcluster

Cor $\mathcal{A}(\mathbb{R}_{\geq 1})$ is closed and bounded.

Pf The generalized associahedron is a polytope. ∎

An application of superunitary regions:
A uniform proof of a previously open conjecture that there are finitely many positive integral friezes, for each Dynkin type.

Def The set of frieze points is

$$\mathcal{A}(\mathbb{Z}_{\geq 1}) := \left\{ \begin{array}{l} \text{ring homomorphisms } p: \mathcal{A} \to \mathbb{R} \text{ such that} \\ p(x) \in \mathbb{Z}_{\geq 1} \text{ for all cluster variables } x \end{array} \right\} \subset \underbrace{\mathcal{A}(\mathbb{R}_{\geq 1})}_{\text{superunitary region}}$$

Cor If $\mathcal{A}$ is a Dynkin type cluster algebra, the set of frieze points is finite.

Pf The homeomorphism $\overbrace{\mathcal{A}(\mathbb{R}_{>0})}^{\text{totally positive region}} \simeq \underbrace{\mathbb{R}^r_{>0}}_{\text{positive orthant}}$

restricts to $\mathcal{A}(\mathbb{Z}_{\geq 1}) \hookrightarrow \underbrace{\mathbb{Z}^r_{\geq 1}}_{\text{pos integral points}}$

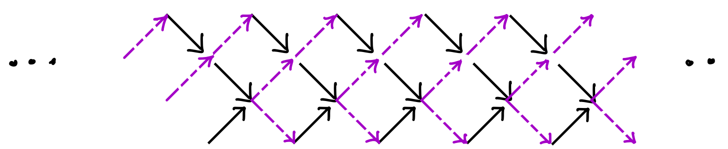
and $\mathcal{A}(\mathbb{Z}_{\geq 1}) \hookrightarrow \underbrace{\mathcal{A}(\mathbb{R}_{\geq 1})}_{\text{superunitary region}}$

Since $\mathcal{A}(\mathbb{R}_{\geq 1})$ is bounded, the set of frieze points is finite. $\square$

What do we mean by positive integral friezes (in this talk)?

<u>Def</u> Q a (valued) Dynkin quiver, e.g. $Q=$
$$\begin{tikzcd}[column sep=small, row sep=small] 4 \arrow[dr] & & \\ & 3 \arrow[dr] & \\ & & 2 \\ & 1 \arrow[ur] & \end{tikzcd}$$
(type A_4)

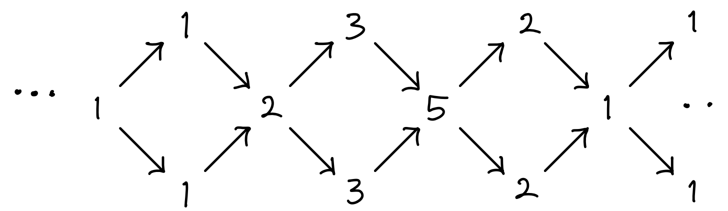
Build the "repetition quiver" $\mathbb{Z}Q$



A positive integral frieze is a function $\mathbb{Z}Q \to \mathbb{Z}_{\geq 1}$ satisfying ...

- [Conway–Coxeter 1970s, type A] for each
$$\begin{tikzcd}[column sep=small, row sep=small] & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}$$
, we have $ad - bc = 1$

E.g. a type A_3 $\mathbb{Z}_{\geq 1}$-frieze



- [Caldero–Chapoton 2006 and Assem–Reutenaur–Smith 2010, in general] for each
$$\begin{tikzcd}[column sep=small, row sep=tiny] & b_1 \arrow[dr] & \\ & b_2 \arrow[r] & d \\ a \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & \\ & b_k \arrow[uur] & \end{tikzcd}$$
, $ad - b_1 \ldots b_k = 1$

friezes $\leftrightarrow$ cluster algebras

<u>Fact</u> If Q is Dynkin, $\underbrace{\mathcal{A}(\mathbb{Z}_{\geq 1})}_{\text{frieze points}} \overset{1-1}{\longleftrightarrow} \mathbb{Z}_{\geq 1}$-friezes of Q

<u>Thm B</u> If Q is Dynkin, there are finitely many $\mathbb{Z}_{\geq 1}$-friezes of Q.

<u>Pf</u> Earlier we said $\mathcal{A}(\mathbb{Z}_{\geq 1})$ is a finite set.

History of proofs, by type	Techniques
✓ Type A Conway–Coxeter 1970s	Polygon triangulations
✓ B, C, D, G_2 Fontaine–Plamondon 2014	Type D triangulations (once-punctured polygon)
✓ E_6, F_4 Cuntz–Plamondon 2018	$\{E_6 \text{ friezes}\} \hookrightarrow \{$2-friezes of height 3$\}$
✓ E_7, E_8 G. — Muller 2022 Conjecture for E_7, E_8 was open until	Uniform proof for all types using compactness of the superunitary region

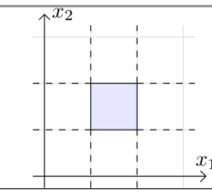
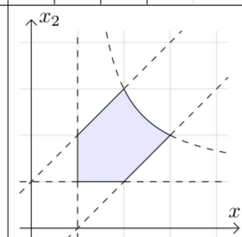
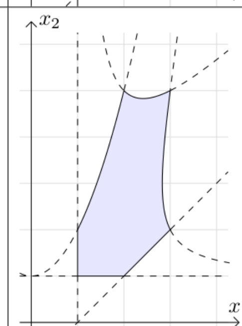
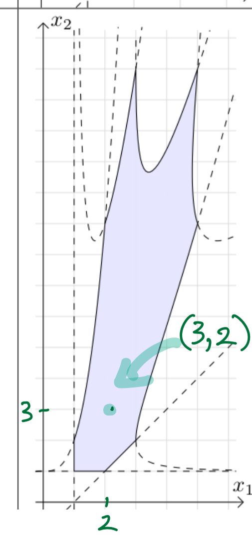
Thm C There is a frieze point in the interior of $\mathcal{A}(\mathbb{R}_{\geqslant 1})$

iff

Q is a union of ...

- type D_n, n not prime
- type E_8
- type B_n, $\sqrt{n+1} \in \mathbb{Z}_{\geqslant 2}$
- type G_2

E.g. In rank 2, there is a frieze point in the interior of $\mathcal{A}(\mathbb{R}_{\geqslant 1})$ iff Q is of type G_2

Quiver	Dynkin type	Superunitary region	Inequalities
$\circ \quad \circ$	$A_1 \times A_1$		$x_1 \geq 1$ $2 \geq x_2$ $2 \geq x_1$ $x_2 \geq 1$
$\circ \longrightarrow \circ$	A_2		$x_1 \geq 1$ $x_1 + 1 \geq x_2$ $x_1 + x_2 + 1 \geq x_1 x_2$ $x_2 + 1 \geq x_1$ $x_2 \geq 1$
$\circ \xrightarrow{(2,1)} \circ$	B_2 or C_2		$x_1 \geq 1$ $x_1^2 + 1 \geq x_2$ $x_1^2 + x_2 + 1 \geq x_1 x_2$ $x_2^2 + 2x_2 + x_1^2 + 1 \geq x_1^2 x_2$ $x_2 + 1 \geq x_1$ $x_2 \geq 1$
$\circ \xrightarrow{(3,1)} \circ$	G_2		$x_1 \geq 1$ $\frac{x_1^3+1}{x_2} \geq 1$ $\frac{x_1^3+x_2+1}{x_2 x_1} \geq 1$ $\frac{x_1^6+3x_2x_1^3+2x_1^3+x_2^3+3x_2^2+3x_2+1}{x_2^2 x_1^3} \geq 1$ $\frac{x_2^3+3x_2^2+3x_2+x_1^3+1}{x_2 x_1^3} \geq 1$ $\frac{x_1^3+x_2^2+2x_2+1}{x_2 x_1^2} \geq 1$ $\frac{x_2+1}{x_1} \geq 1$ $x_2 \geq 1$

THANK
YOU

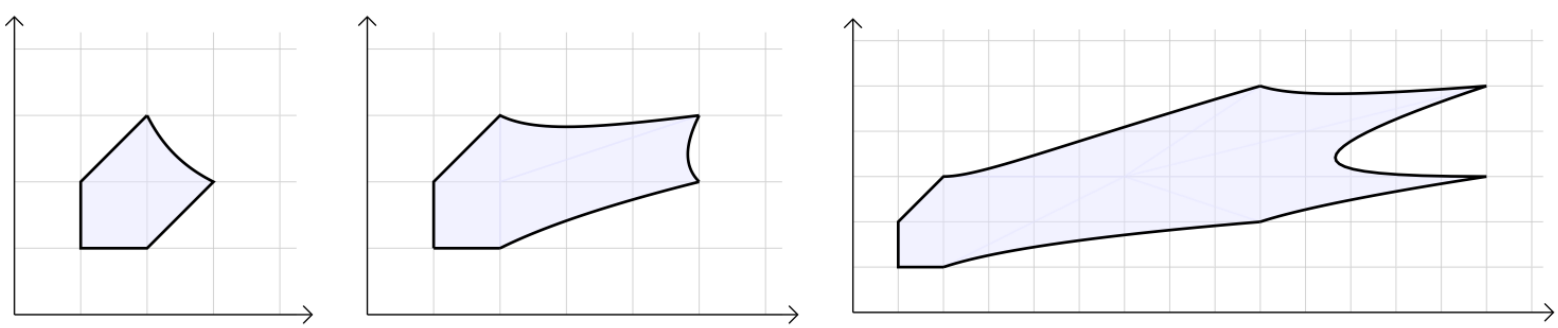


FIGURE 1. The superunitary regions of types A_2, B_2/C_2, and G_2 (embedded in $\mathbb{R}^2_{>0}$)