Short Communication:

Triangulations & maximal almost rigid modules

Emily Gunawan (University of Oklahoma)

Jt. w/ Emily Barnard, Raquel Coelho Simões, & Ralf Schiffler

Maurice Auslander
Distinguished Lectures and
International Conference

Sunday, October 30, 2022

Gentle algebras

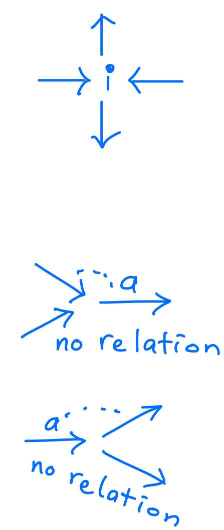
Def A finite-dimensional algebra $A = \mathbb{k}Q/I$ is <u>gentle</u> if :

(G1) $\forall$ vertex i of Q, $\exists$ at most 2 arrows starting at i
$\exists$ at most 2 arrows ending at i

(G2) I is generated by paths of length 2

(G3) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b s.t $ba \notin I$
$\exists$ at most 1 arrow c s.t $ac \notin I$

(G4) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b' s.t $b'a \in I$
$\exists$ at most 1 arrow c' s.t $ac' \in I$



<u>Ex</u>

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$\mathbb{k}Q$ <u>Not</u> gentle

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

$\mathbb{k}Q/I$ is gentle

$$Q = \begin{tikzcd} & 4 \arrow[dl, "c"'] & \\ 3 & 2 \arrow[l, "d"'] \arrow[r, "a"'] & 1 \arrow[ul, "b"'] \end{tikzcd}$$

$I = \langle ab, ca \rangle$

$\mathbb{k}Q/I$ is also gentle

<u>Recall def</u> $T \in \mathrm{mod}(A)$ is <u>tilting</u> if

(T1) T has $|Q_0|$ non-isomorphic summands
\# of vertices of Q

(T2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$

(T3) T has projective dimension at most 1.

<u>Def</u> $A = \mathbb{k}Q/I$ finite representation type gentle algebra.

$T \in \mathrm{mod}(A)$ is <u>maximal almost rigid</u> (mar) if

(M1) T has $|Q_0|$ + $|Q_1|$ non-isomorphic summands
\# of vertices of Q \# of arrows of Q

(M2) For each pair A, B of summands of T,
if $0 \to B \to E \to A \to 0$ is a short exact sequence,
then $E \cong B \oplus A$ or E is indecomposable

called <u>almost rigid</u>

<u>Rem</u> (M1) can be replaced with:
"T is maximal with respect to (M2)"

[Opper – Plamondon – Schroll 2018]
[Baur – Coelho Simões 2018] + extra marked points $\ast$

$A = \Bbbk Q / I$ is a gentle algebra iff

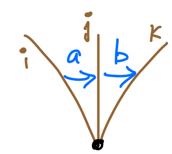
A comes from a 4-tuple (S, M, P, R)

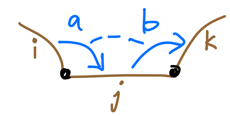
- S: oriented surface
- M: marked points in ∂S
- P: arcs in (S, M)
- R: marked points in S

$\bullet \in M$

$\ast \in R$

"Rules":

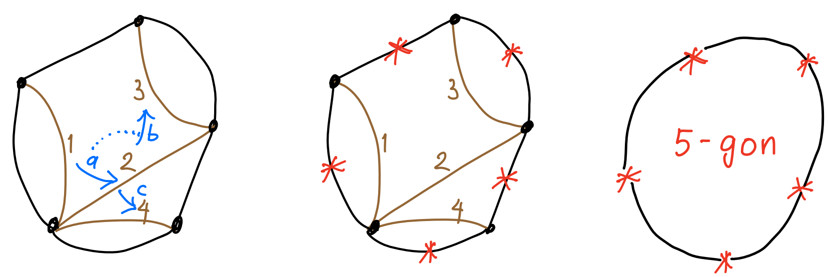
$i \xrightarrow{a} j \xrightarrow{b} k$ $\quad ab \notin I$ corresponds to configuration  of P

$i \xrightarrow{a} j \xrightarrow{b} k$ $\quad ab \in I$ corresponds to configuration  of P

Ex

$Q = 1 \xrightarrow{a} 2$, $2 \xrightarrow{b} 3$, $2 \xrightarrow{c} 4$

$I = \langle ab \rangle$



$(S, M, P) \Rightarrow (S, M, P, R) \Rightarrow (S, R)$

Thm1 Let $A = \mathbb{k}Q/I$ be a finite representation type gentle algebra.

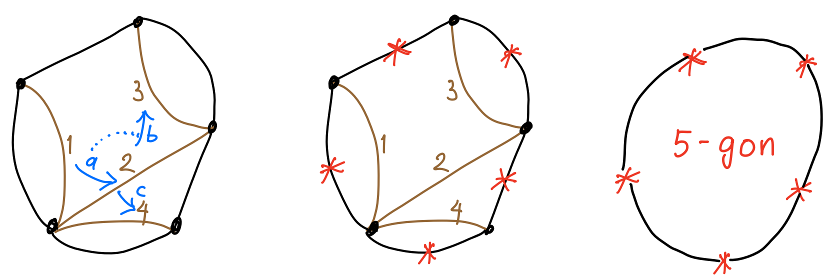
$$\left\{ \underset{\text{mar}(A)}{\text{mar modules}} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{"permissible" ideal triangulations of } (S, R) \\ \text{including boundary edges} \end{array} \right\}$$

Ex

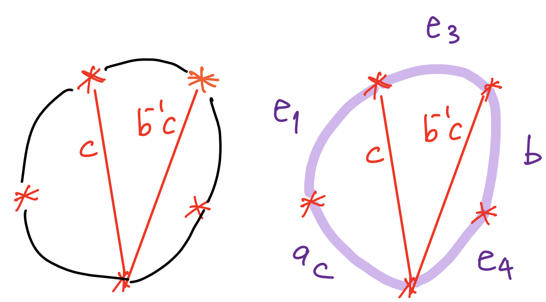
$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

Each T in $\text{mar}(A)$ has $\underset{4}{|Q_0|} \underset{+}{+} \underset{3}{|Q_1|}$ summands



$(S, M, P) \to (S, M, P, R) \to (S, R)$



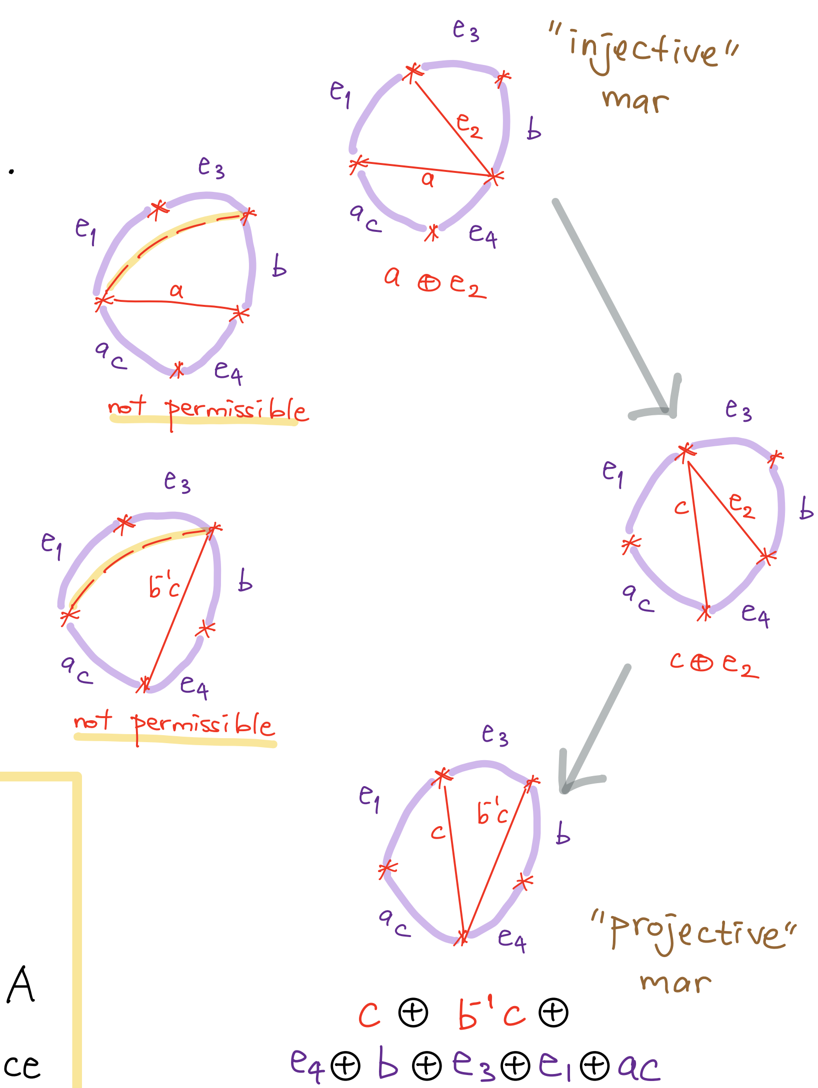
$c \oplus b^{-1}c \oplus \underbrace{e_4 \oplus b \oplus e_3 \oplus e_1 \oplus ac}_{} \in \text{mar}(A)$

These 5 summands are required in every $T \in \text{mar}(A)$

Def/Prop 2 The oriented flip graph of $\operatorname{mar}(\Bbbk Q/I)$: arrows are "positive" diagonal flips

Ex The flip graph of $\operatorname{mar}(\Bbbk Q/I)$

for $Q = 1 \xrightarrow{a} 2$, $2 \xrightarrow{b} 3$, $2 \xrightarrow{c} 4$ $\quad I = \langle ab \rangle$ is ...



> [Barnard – G. – Meehan – Schiffler 2019]
> In type A, the flip graph of $\operatorname{mar}(\Bbbk Q)$ is an oriented exchange graph of a type A cluster algebra & a Tamari/Cambrian lattice

$\underline{\text{Thm3}}$

Construct a new gentle algebra

$$\overline{A} := \Bbbk\overline{Q}/\overline{I} \quad \text{where} \quad |\overline{Q}_0| = |Q_0| + |Q_1| .$$

Then $M \in \operatorname{mar}(A) \implies \operatorname{End}_A(M) \cong \operatorname{End}_{\overline{A}}(T)$

where T is tilting in $\operatorname{mod}(\overline{A})$.

$$Q = \begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[dr, "c"'] & 3 \\ & & 4 \end{tikzcd} \qquad I = \langle ab \rangle$$

$$\overline{Q} = \begin{tikzcd} 1 \arrow[r, "a_1"] & a \arrow[r, "a_2"] & 2 \arrow[r, "b_1"] \arrow[dr] & b \arrow[r] & 3 \\ & & & c \arrow[r] & 4 \end{tikzcd} \qquad \overline{I} = \langle a_2 b_1 \rangle$$

$A = \Bbbk Q / I$

$$\begin{tikzcd}[column sep=small] & & & 124 \arrow[dr] & & \\ 3 \arrow[dr] & & 24 \arrow[ur] \arrow[dr] & & 12 \arrow[dr] & \\ & 234 \arrow[ur] \arrow[dr] & & 2 \arrow[ur] & & 1 \\ 4 \arrow[ur] & & 23 \arrow[ur] & & & \end{tikzcd}$$

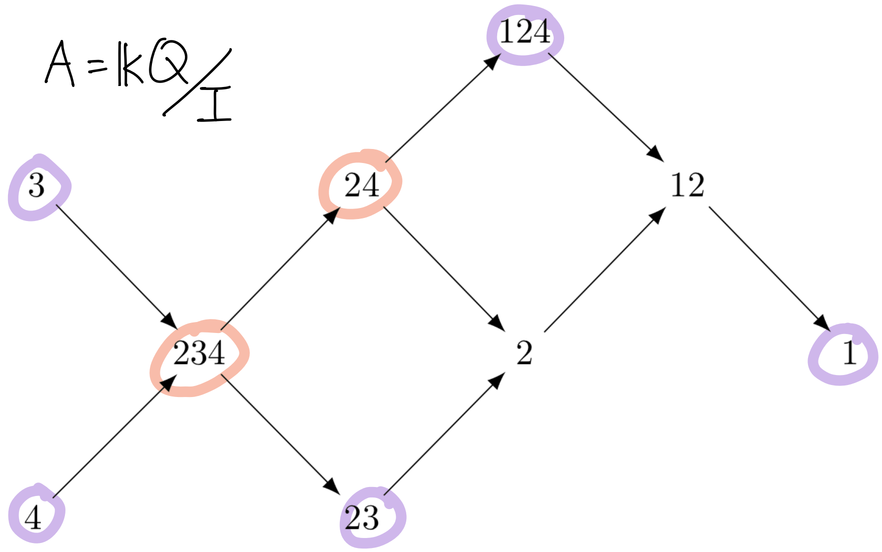
$\overline{A} = \Bbbk\overline{Q}/\overline{I}$

$$\begin{tikzcd}[column sep=small] & & & 124 \arrow[dr] & & \\ 3 \arrow[dr] & & 24 \arrow[ur] \arrow[dr] & & 12 \arrow[dr] & \\ & 234 \arrow[ur] \arrow[dr] & & 2 \arrow[ur] & & 1 \\ 4 \arrow[ur] & & 23 \arrow[ur] & & & \end{tikzcd}$$

$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad I = \langle ab \rangle$$

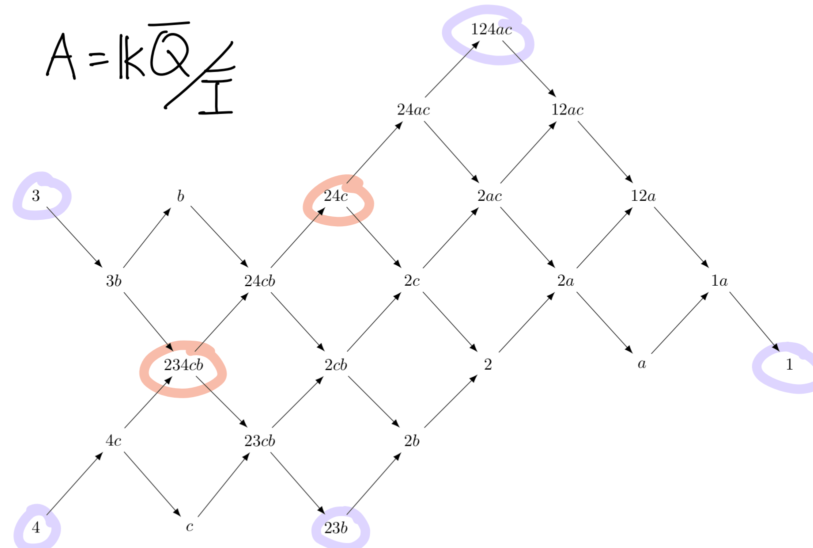
$$\overline{Q} = \begin{tikzcd} & & & b \arrow[r] & 3 \\ 1 \arrow[r, "a_1"] & a \arrow[r, "a_2"] & 2 \arrow[ur, "b_1"] \arrow[dr] & & \\ & & & c \arrow[r] & 4 \end{tikzcd} \qquad \overline{I} = \langle a_2 b_1 \rangle$$

$A = \mathbb{k}Q/I$



$M \in \mathrm{mar}(A)$

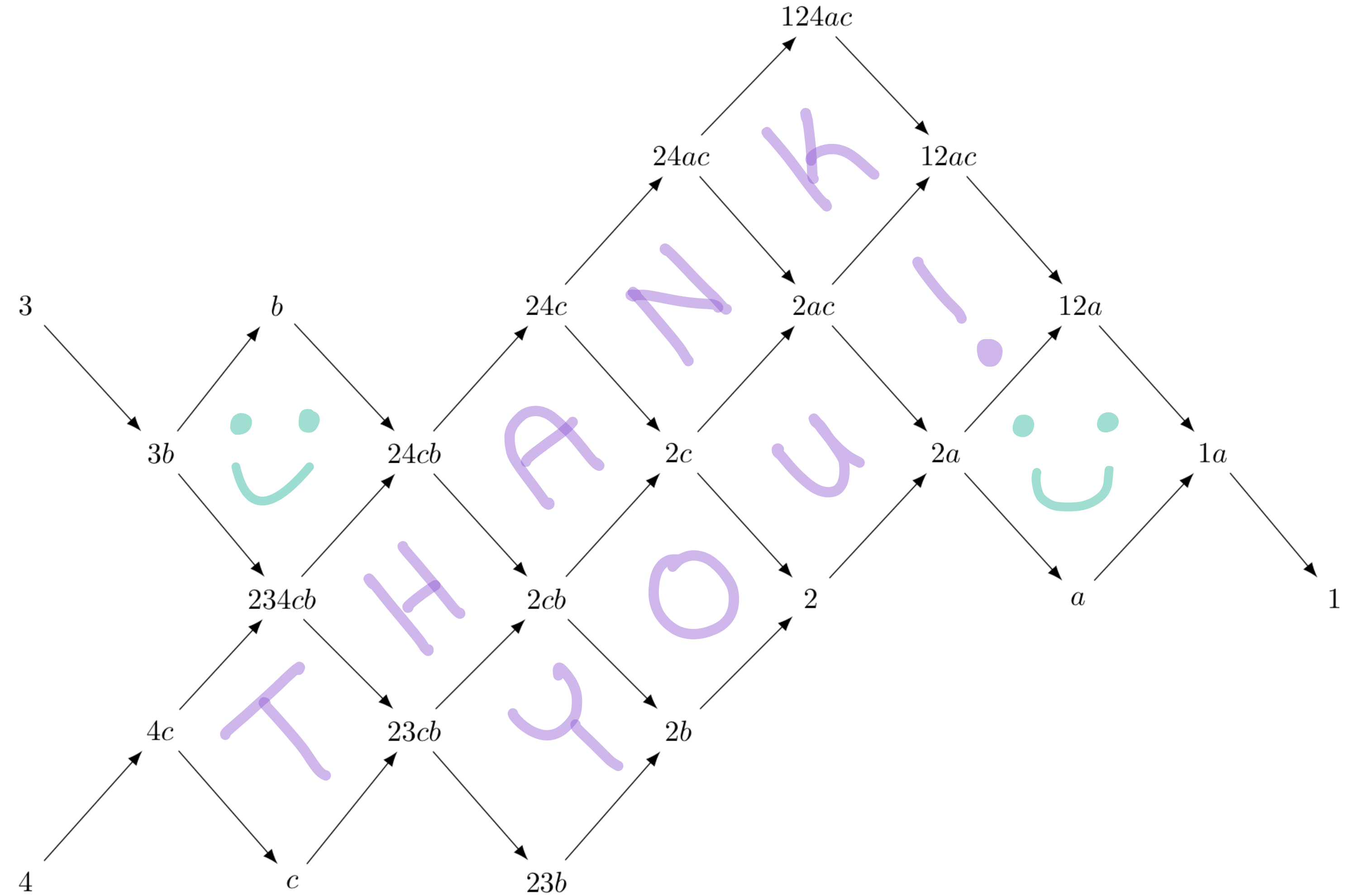
$A = \mathbb{k}\overline{Q}/\overline{I}$



T is a tilting module over $\overline{A}$

mar over A tilting over $\overline{A}$

Thm 3 $\mathrm{End}_A(M) \cong \mathrm{End}_{\overline{A}}(T)$



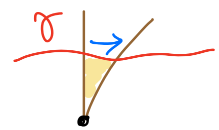
124ac
24ac
12ac
3
b
24c
2ac
12a
3b
24cb
2c
2a
1a
234cb
2cb
2
a
1
4c
23cb
2b
4
c
23b
THANK YOU!

What is a permissible arc? (extra)

string modules of $\Bbbk Q/I \overset{1-1}{\longleftrightarrow}$ permissible arcs γ in S:

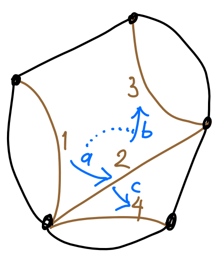
(i) endpoints are in R

(ii) each pair of consecutive crossings of γ and P corresponds to an arrow of Q



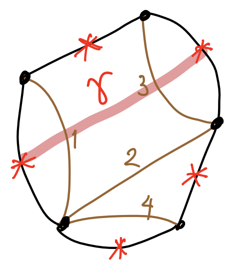
(locally cuts up a triangle)

Ex (S, M, P)

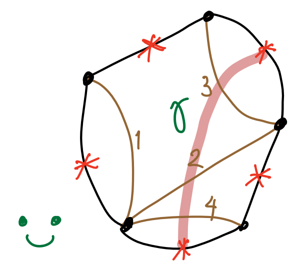


$$Q = 1 \xrightarrow{a} 2 \quad 2 \xrightarrow{b} 3 \quad 2 \xrightarrow{c} 4$$

$I = \langle ab \rangle$

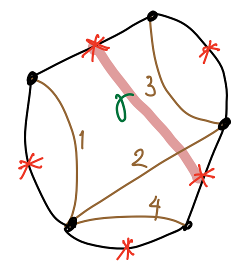


- γ is not permissible
- Consecutive crossings arc 1, arc 3 do not correspond to an arrow of Q



permissible γ

$\leftrightarrow$ string $b^{-1}c$ or $c^{-1}b$



permissible γ

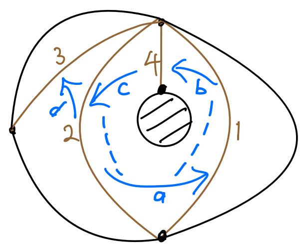
$\leftrightarrow$ trivial string e_2

Example (extra)

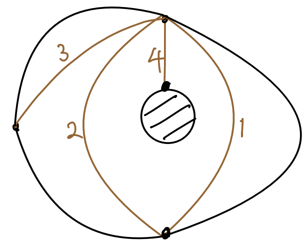
$$Q = \begin{tikzcd} & 4 \arrow[dl, "c"'] & \\ 3 & 2 \arrow[l, "d"'] \arrow[r, "a"'] & 1 \arrow[ul, "b"'] \end{tikzcd}$$

$I = \langle ab, ca \rangle$

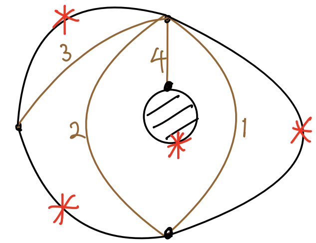
(S, M, P)



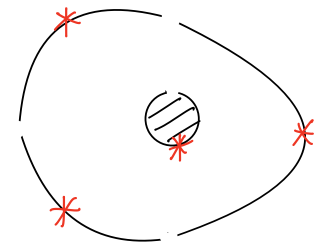
(S, M, P)



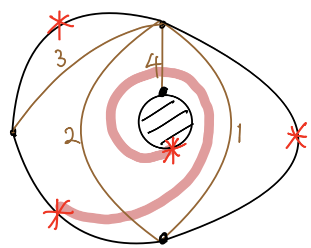
(S, M, P, R)



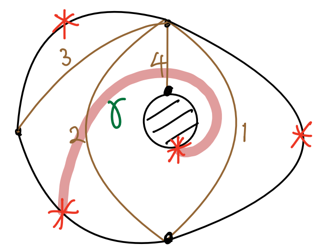
(S, R)



An annulus with
3 points on one bdry
& 1 point on the other

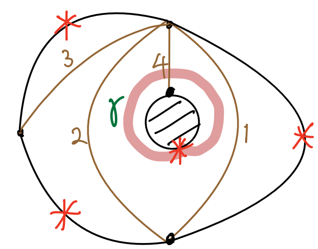


Not permissible
because
the crossings
w/ P at
arc 2, arc 4
do not
correspond
to an arrow in Q



permissible

$\gamma \longleftrightarrow$ string $4 \xrightarrow{c} 2$



permissible

$\gamma \longleftrightarrow$ trivial string e_4