Triangulations & maximal almost rigid modules

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Representations of Algebras
and
Related Combinatorics

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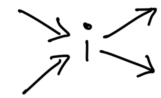
Gentle bound path algebras

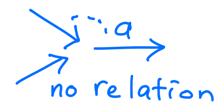
<u>Def</u> A finite-dimensional algebra $A = \mathbb{k}Q/I$ is <u>gentle</u> if :



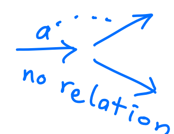
(G1) $\forall$ vertex i of Q, $\exists$ at most 2 arrows starting at i
$\exists$ at most 2 arrows ending at i

(G2) I is generated by paths of length 2

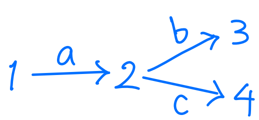
(G3) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b s.t $ba \notin I$
$\exists$ at most 1 arrow c s.t $ac \notin I$



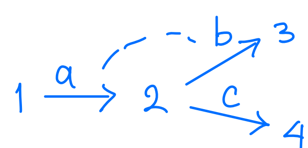
(G4) $\forall$ arrow a of Q, $\exists$ at most 1 arrow b' s.t $b'a \in I$
$\exists$ at most 1 arrow c' s.t $ac' \in I$



<u>Ex</u>

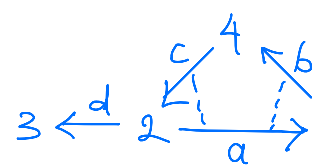
$Q =$ 

$\mathbb{k}Q$ <u>Not</u> gentle

$Q =$ 

$I = \langle ab \rangle$

$\mathbb{k}Q/I$ is gentle

$Q =$ 

$I = \langle ab, ca \rangle$

$\mathbb{k}Q/I$ is also gentle

String modules

A <u>string</u> ω is a walk along the arrows in $Q_1 \cup \underbrace{Q_1^{-1}}_{\{\text{opposite arrow } \alpha^{-1} \mid \alpha \in Q_1\}}$

with
- no backtrack aa^{-1} or $a^{-1}a$
- no subwalk v with $v \in I$ or $v^{-1} \in I$

  (no going through relations)

<u>Ex</u>

$$Q = \begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad I = \langle ab \rangle$$

- ac and $b^{-1}c$ are strings, ab is <u>not</u> a string

$$\begin{tikzcd}[row sep=small] 1 \arrow[r, "a"] & 2 \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & 3 \\ 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & 4 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] & \end{tikzcd}$$

- e_1, e_2, e_3, e_4 are <u>trivial strings</u>

[Butler–Ringel 1987]

- ω string $\longleftrightarrow$ $M(\omega)$ string module
- If $\Bbbk Q/I$ is a <u>finite representation type</u> gentle algebra,

  finitely many indecomposable modules

  indecomposable modules are string modules.

<u>Recall def</u> $T \in \mathrm{mod}(A)$ is <u>tilting</u> if

(T1) T has $\underset{\text{vertices of } Q}{|Q_0|}$ non-isomorphic summands

(T2) For each pair A, B of summands of T,

if $0 \to B \to E \to A \to 0$ is a short exact sequence, then $E \cong B \oplus A$

(T3) T has projective dimension at most 1.

<u>Def</u> $A = \Bbbk Q/I$ finite representation type gentle algebra.

$T \in \mathrm{mod}(A)$ is <u>maximal almost rigid</u> (mar) if

(M1) T has $\underset{\text{vertices of } Q}{|Q_0|} + \underset{\text{arrows of } Q}{|Q_1|}$ non-isomorphic summands

(M2) For each pair A, B of summands of T,

if $0 \to B \to E \to A \to 0$ is a short exact sequence,

then $E \cong B \oplus A$ or E is indecomposable

<u>Rem</u> (M1) can be replaced with :

"T is maximal with respect to (M2)"

MAIN RESULTS

$A = {}^{\mathbb{k}Q}/_{I}$ gentle & finite representation type.

1. $\{$ mar modules $\} \longleftrightarrow \{$ "permissible" ideal triangulations of a marked surface $\}$

   $\mathrm{mar}(A) :=$

2. Define an oriented flip graph of $\mathrm{mar}(A)$ such that

   the "projective" mar is a sink &

   the "injective" mar is a source

3. Construct a new gentle algebra

   $\bar{A} := {}^{\mathbb{k}\bar{Q}}/_{\bar{I}}$ where $|\bar{Q}_0| = |Q_0| + |Q_1|$.

Then $M \in \mathrm{mar}(A) \Rightarrow \mathrm{End}_A(M) \cong \mathrm{End}_{\bar{A}}(T)$ for some tilting module T in $\mathrm{mod}(\bar{A})$

[Opper – Plamondon – Schroll 2018]
[Baur – Coelho Simões 2018] + slight modification

$A = \Bbbk Q / I$ is a gentle algebra iff

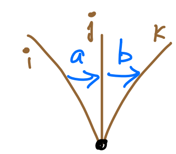
A comes from a 4-tuple (S, M, P, R)

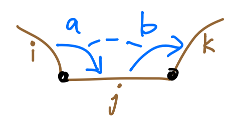
- S: oriented surface
- M: marked points in ∂S
- P: arcs in (S, M)
- R: marked points in S

$\bullet \in M$

$* \in R$

"Rules":

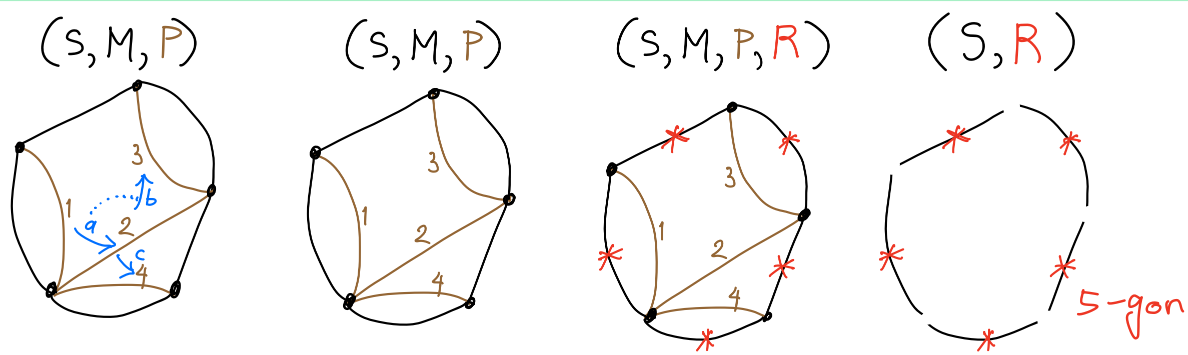
$i \xrightarrow{a} j \xrightarrow{b} k$ $\quad ab \notin I$ corresponds to configuration  of P

$i \xrightarrow{a} j \xrightarrow{b} k$ $\quad ab \in I$ corresponds to configuration  of P

Ex

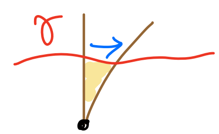
$$Q = \begin{array}{l} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \\ \phantom{1 \xrightarrow{a}} 2 \xrightarrow{c} 4 \end{array}$$

$I = \langle ab \rangle$



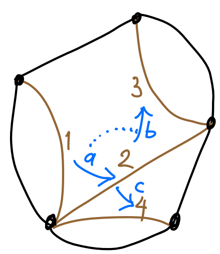
string modules of $\Bbbk Q/I$ $\overset{1-1}{\longleftrightarrow}$ <u>permissible</u> arcs γ in S:

* endpoints are in R
* each pair of consecutive crossings of γ and P corresponds to an arrow of Q



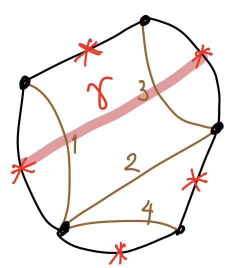
(locally cuts up a triangle)

<u>Ex</u> (S, M, P)



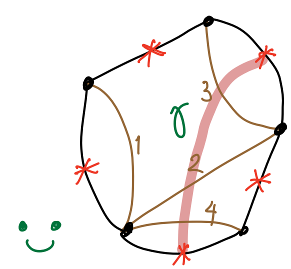
$$Q = \begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$I = \langle ab \rangle$

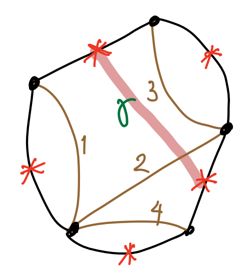


• γ is not permissible

• Consecutive crossings arc 1, arc 3 do not correspond to an arrow of Q



permissible γ $\leftrightarrow$ string $b^{-1}c$ or $c^{-1}b$



permissible γ $\leftrightarrow$ trivial string e_2

Thm 1 $\left\{\begin{array}{l}\text{mar modules} \\ \text{mar}(A)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{permissible ideal triangulations of } (S, R) \\ \text{including boundary edges}\end{array}\right\}$

Ex

$Q = 1 \xrightarrow{a} 2$, $2 \xrightarrow{b} 3$, $2 \xrightarrow{c} 4$

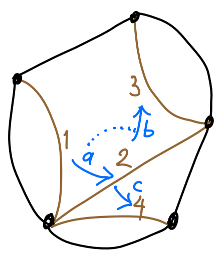
$I = \langle ab \rangle$

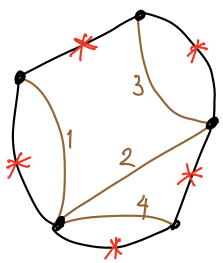
Each T in mar(A)

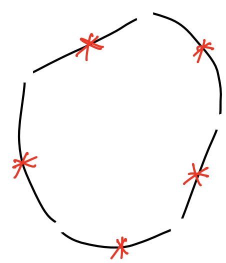
has $|Q_0| + |Q_1|$ summands

$4 + 3$

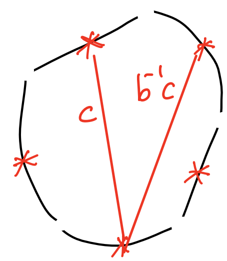
(S, M, P) (S, M, P, R) (S, R)

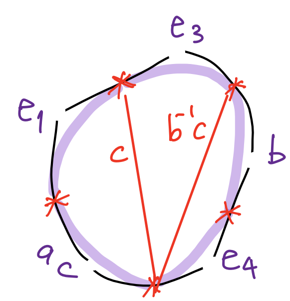






5-gon





$c \oplus b'c \oplus \underbrace{e_4 \oplus b \oplus e_3 \oplus e_1 \oplus ac}_{} \in \text{mar}(A)$

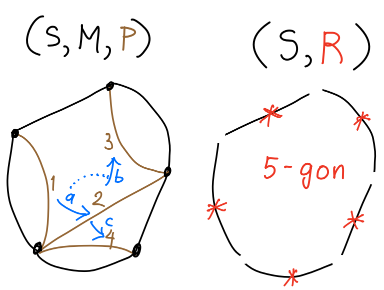
These 5 summands are required in every $T \in \text{mar}(A)$

Def/Prop 2 The oriented flip graph of $\operatorname{mar}(\Bbbk Q/I)$:

arrows $\leftrightarrow$ "positive" diagonal flips

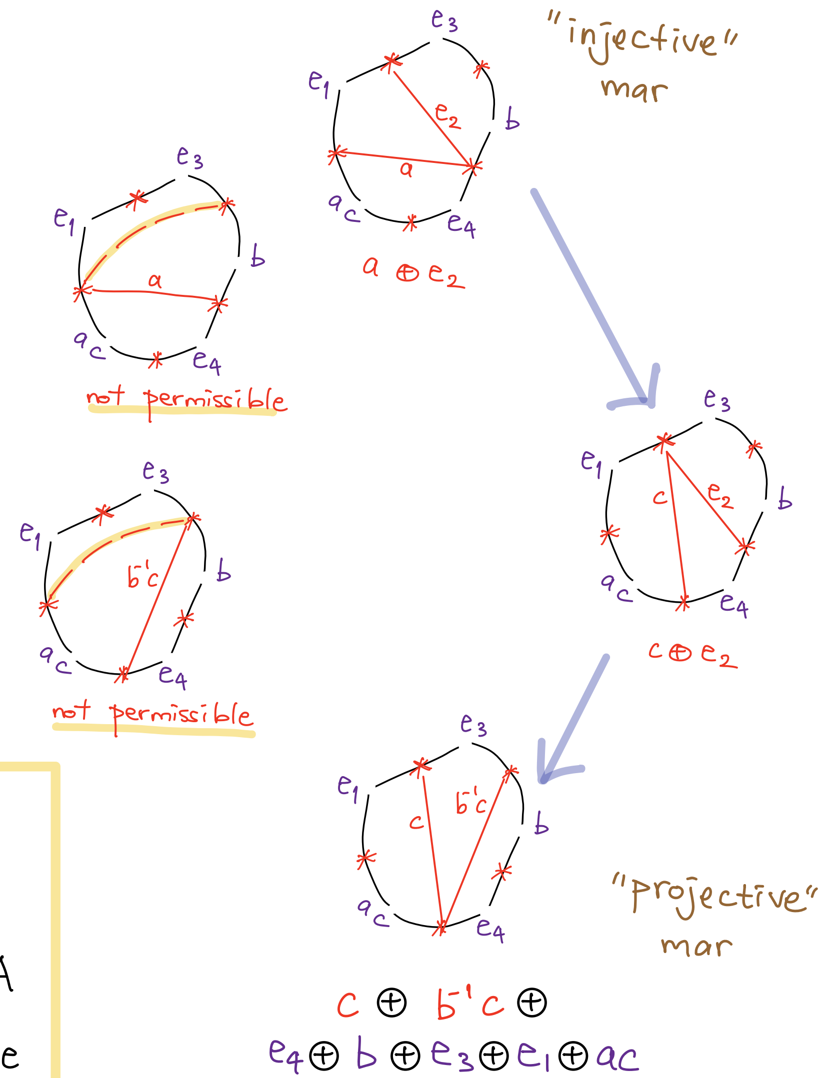
Ex Flip graph of $\operatorname{mar}(\Bbbk Q/I)$

for $Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$, $2 \xrightarrow{c} 4$ $\qquad I = \langle ab \rangle$



[Barnard - G. - Meehan - Schiffler 2019]

In type A, the flip graph of $\operatorname{mar}(\Bbbk Q)$ is an oriented exchange graph of a type A cluster algebra & a Tamari/Cambrian lattice



$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3,\quad 2 \xrightarrow{c} 4$

$I = \langle ab \rangle$

$i \xrightarrow{\alpha} j \;\Rightarrow\; i \xrightarrow{\alpha_1} \alpha \xrightarrow{\alpha_2} j$

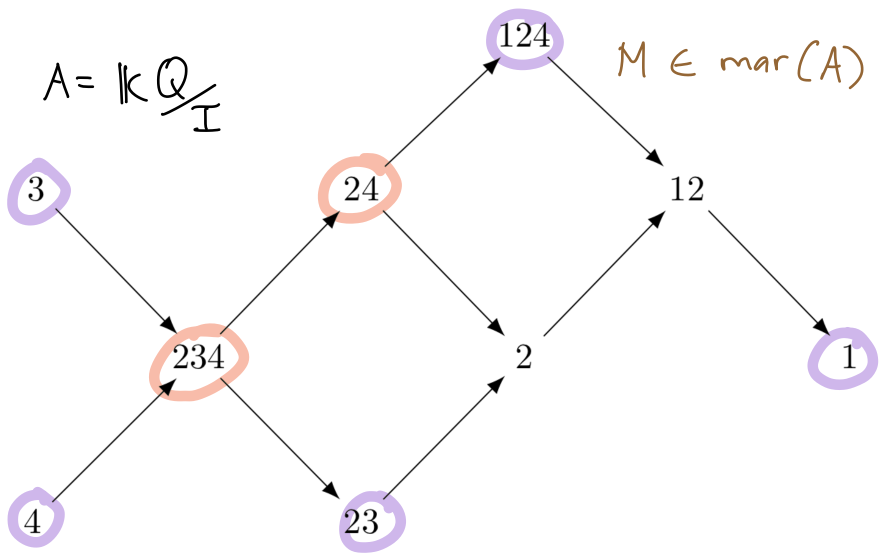
$i \xrightarrow{\alpha} j \xrightarrow{\beta} k,\ \alpha\beta \in I \;\Rightarrow\; i \xrightarrow{\alpha_1} \alpha \xrightarrow{\alpha_2} j \xrightarrow{\beta_1} \beta \xrightarrow{\beta_2} k,\ \alpha_2\beta_1 \in \overline{I}$

$\overline{Q} = 1 \xrightarrow{a_1} a \xrightarrow{a_2} 2 \xrightarrow{b_1} b \to 3,\quad 2 \to c \to 4$

$\overline{I} = \langle a_2 b_1 \rangle$

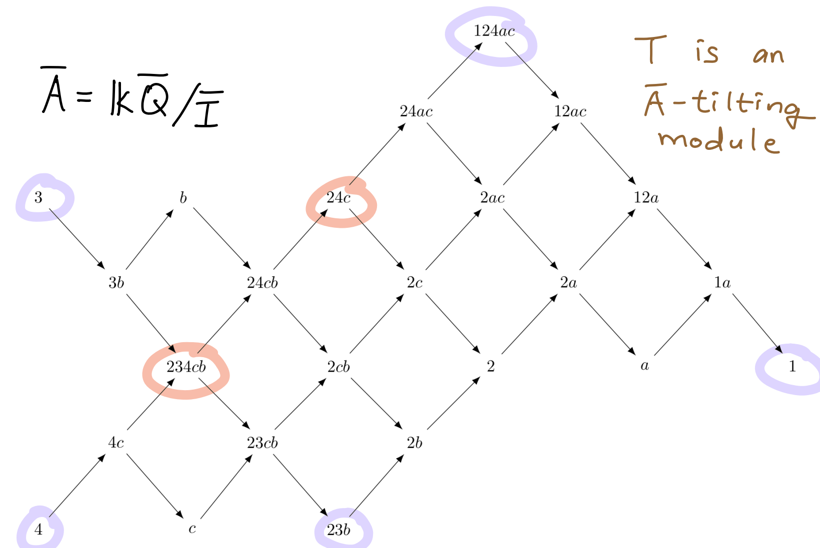
$A = \Bbbk Q / I$

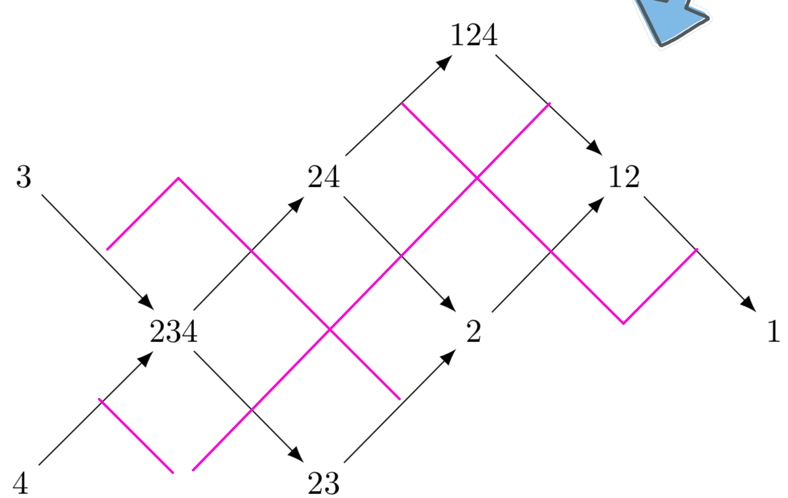
$M \in \operatorname{mar}(A)$



$\overline{A} = \Bbbk \overline{Q} / \overline{I}$

T is an $\overline{A}$-tilting module





Thm 3 $\operatorname{End}_A M \cong \operatorname{End}_{\overline{A}} T$

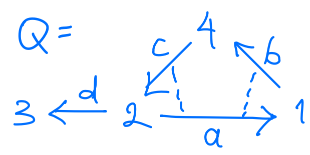
$M \in \operatorname{mar}(A)$, T tilting in $\overline{A}$

$$
\begin{tikzcd}
 & & & & & & 124ac \arrow[dr] & & & & \\
 & & & & & 24ac \arrow[ur] \arrow[dr] & & 12ac \arrow[dr] & & & \\
3 \arrow[dr] & & b \arrow[dr] & & 24c \arrow[ur] \arrow[dr] & & 2ac \arrow[ur] \arrow[dr] & & 12a \arrow[dr] & & \\
 & 3b \arrow[ur] \arrow[dr] & & 24cb \arrow[ur] \arrow[dr] & & 2c \arrow[ur] \arrow[dr] & & 2a \arrow[ur] \arrow[dr] & & 1a \arrow[dr] & \\
 & & 234cb \arrow[ur] \arrow[dr] & & 2cb \arrow[ur] \arrow[dr] & & 2 \arrow[ur] & & a \arrow[ur] & & 1 \\
 & 4c \arrow[ur] \arrow[dr] & & 23cb \arrow[ur] \arrow[dr] & & 2b \arrow[ur] & & & & & \\
4 \arrow[ur] & & c \arrow[ur] & & 23b \arrow[ur] & & & & & &
\end{tikzcd}
$$

THANK YOU!

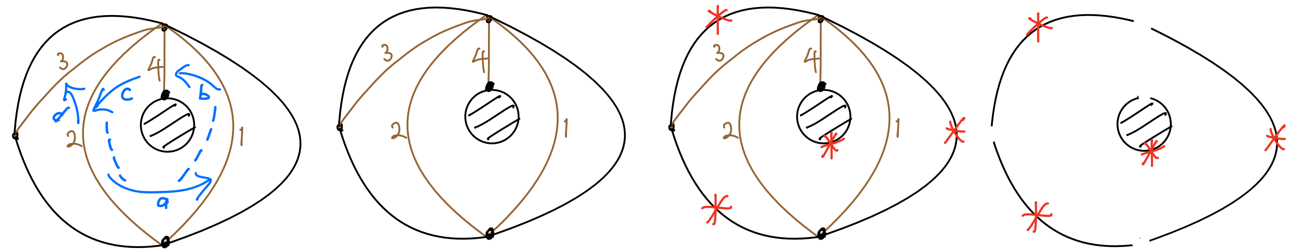
Example (extra)

$Q =$

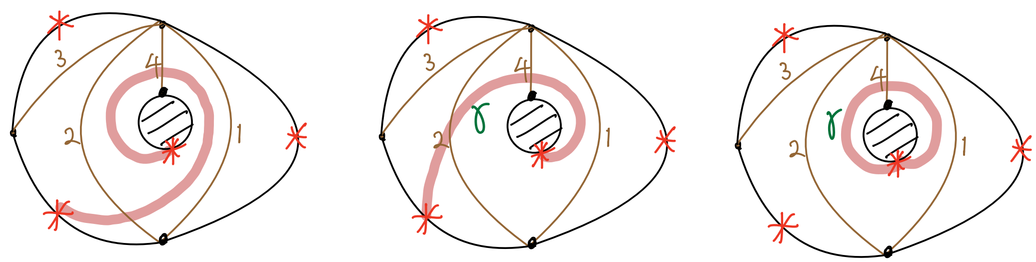


$I = \langle ab, ca \rangle$

(S, M, P) (S, M, P) (S, M, P, R) (S, R)



An annulus with 3 points on one bdry & 1 point on the other



Not permissible because the crossings w/ P at arc 2, arc 4 do not correspond to an arrow in Q

permissible $\gamma \longleftrightarrow$ string $4 \xrightarrow{c} 2$

permissible $\gamma \longleftrightarrow$ trivial string e_4

Oriented flip graph (extra)

<u>Prop</u> Sum of $\{P(i)\} \oplus \{$summands of rad $P(i)\} \oplus$
$\{$required summands$\}$ is a mar

<u>Idea:</u> Define an oriented flip graph so that
the above "projective" mar is a sink & the "injective" mar is a source.

<u>Lemma</u> If $T_1, T_2 \in \operatorname{mar}(A)$ s.t $T_1/M_1 \cong T_2/M_2$
then $\exists$ a unique overlap extension between T_1 and T_2.
The middle term lies in T_1/M_1.

<u>Def</u> Define the flip graph on $\operatorname{mar}(A)$ by $T_2 \swarrow T_1$
whenever such condition is satisfied.

<u>Prop</u> The "projective" mar is a sink &
the "injective" mar is a source
in this flip graph of $\operatorname{mar}(A)$.