Box-ball systems and Robinson–Schensted–Knuth tableaux

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Solitary waves (solitons)

Scott Russell's first encounter of solitary waves at the Union Canal:

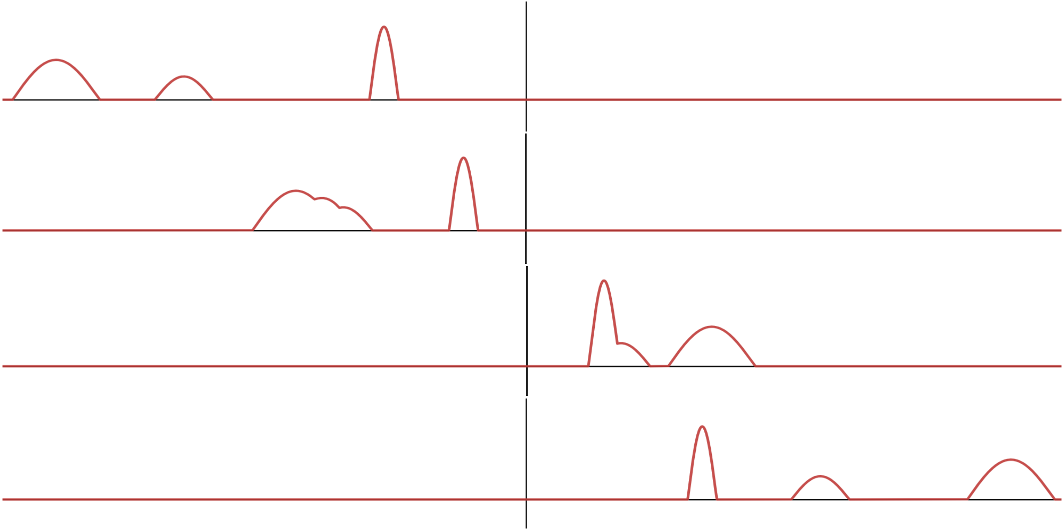
'I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently **without change of form or diminution of speed**. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.'

Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, July 1995

Credit: ma.hw.ac.uk/solitons/press.html

Solitary waves

(Desmos link by D. Zeng)



Multicolor box-ball system (BBS), Takahashi 1993

A *box-ball system* (BBS) is a dynamical system of BBS configurations.

- At each configuration, balls are labeled by numbers 1 through n in an infinite strip of boxes.
- Each box can fit at most one ball.

Example

A possible BBS configuration:

$\cdots$	4	5	2	3	6	1									$\cdots$

Box-ball move (from $t = 0$ to $t = 1$)

Balls take turns jumping to the first empty box to the right, starting with the smallest-numbered ball.

$t = 0$	$\cdots$	4	5	2	3	6	1				$\cdots$
	$\cdots$	4	5	2	3	6		1			$\cdots$
	$\cdots$	4	5		3	6	2	1			$\cdots$
	$\cdots$	4	5			6	2	1	3		$\cdots$
	$\cdots$		5	4		6	2	1	3		$\cdots$
	$\cdots$			4	5	6	2	1	3		$\cdots$
$t = 1$	$\cdots$			4	5		2	1	3	6	$\cdots$

Box-ball moves ($t = 0$ through $t = 5$)

$t = 0$	$\cdots$	4	5	2	3	6	1																$\cdots$
$t = 1$	$\cdots$			4	5		2	1	3	6													$\cdots$
$t = 2$	$\cdots$					4	5	2			1	3	6										$\cdots$
$t = 3$	$\cdots$							4	2	5				1	3	6							$\cdots$
$t = 4$	$\cdots$								4		2	5					1	3	6				$\cdots$
$t = 5$	$\cdots$									4			2	5						1	3	6	$\cdots$

Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to its length and is preserved by all future BBS moves.

Example

The strings 4, 25, and 136 are solitons:

$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

After a finite number of BBS moves, the system reaches a *steady state* where:

- the system is decomposed into solitons, i.e., each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

Tableaux (English notation)

Definition

- A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

1	2	4
3	5	
6	7	

1	3	6
2	5	
4		

1	3	4
2		
5		
6		

Not a tableau:

1	2	
3	5	4

Nonstandard Tableau:

1	2	3
5	6	7
4		

Soliton decomposition

Definition

- Let S_n be the symmetric group on n elements. We represent permutations of S_n in *one-line notation* as $w = w(1)w(2)\cdots w(n)$.
- To construct *soliton decomposition* $\mathrm{SD}(w)$ of a permutation w, start with w in one-line notation, run BBS moves until we get to a steady state, then stack solitons so that the rightmost soliton is placed on the first row, the soliton to its left is placed on the second row, and so on.

Example

$t=3$							4	2	5				1	3	6						
$t=4$								4		2	5					1	3	6			

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \text{ with shape } (3,2,1).$$

RSK algorithm

The Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $w =$ **452361**. $\mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ and $\mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$.

RSK algorithm example

Let $w = \mathbf{452361}$.

$$\text{P}: \quad \mathbf{4} \qquad \begin{array}{cc} 4 & \mathbf{5} \end{array} \qquad \begin{array}{cc} \mathbf{2} & 5 \\ 4 & \end{array} \qquad \begin{array}{cc} 2 & \mathbf{3} \\ 4 & 5 \end{array} \qquad \begin{array}{ccc} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{array} \qquad \begin{array}{ccc} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \qquad \text{P}(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$\text{Q}: \quad \mathbf{1} \qquad \begin{array}{cc} 1 & \mathbf{2} \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \mathbf{3} & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & \mathbf{4} \end{array} \qquad \begin{array}{ccc} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{array} \qquad \text{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Insertion and bumping rule for P

- Insert x into the first row of P.
- If x is larger than every element in the first row, add x to the end of the first row.
- If not, replace the smallest number larger than x in row 1 with x. Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

The Q tableau determines the dynamics of a box-ball system

Theorem (SUMRY 2021)

If $\mathrm{Q}(v) = \mathrm{Q}(w)$, then the box-ball systems of v and w are identical if we ignore the ball labels, in particular:

- v and w first reach steady state at the same time, and
- the soliton decompositions of v and w have the same shape

Example

$$v = 21435 \text{ and } w = 31425$$

$$\mathrm{Q}(v) = \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$$

Both v and w first reach steady state at $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

Questions (steady-state time)

The time when a permutation w first reaches steady state is called the *steady-state time* of w.

- Given a Q-tableau, find a formula to compute the steady-state time for all permutations in the Q-tableau class.
- Find an upper bound for steady-state time.

L-shaped soliton decompositions

Theorem (SUMRY 2021)

If a permutation has an L-shaped soliton decomposition $\mathrm{SD} = \begin{array}{|c|c|c|l} \hline 1 & & & \cdots \\ \hline & \multicolumn{3}{l}{} \\ \cline{1-1} & \multicolumn{3}{l}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \end{array}$,

then its steady-state time is either $t = 0$ or $t = 1$.

Example

Such permutations include noncrossing involutions and column reading words of standard tableaux.

Both $v = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$\mathrm{SD}(v) = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$v = 21435 = (12)(34)$ and $w = 31425$ is the column reading word of $\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$.

Maximum steady-state time

Theorem (UConn 2020)

If $n \geq 5$ and

$$Q(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1} \end{array},$$

then the steady-state time of w is $n-3$.

Conjecture

For $n \geq 4$, the steady-state time of a permutation in S_n is at most $n-3$.

Partial Result (SUMRY 2021)

If the shape of $Q(w)$ is $(n-3, 2, 1)$, the steady-state time is at most $n-3$.

Box-Ball System Example ($t = 0$ through 5)

Let $w = 452361$. Then $\mathrm{Q}(w) =$

1	2	5
3	4	
6		

and the steady-state time of w is $3 = n - 3$.

$t = 0$	4	5	2	3	6	1															
$t = 1$			4	5		2	1	3	6												
$t = 2$					4	5	2			1	3	6									
$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

Questions (soliton decomposition)

- When is the soliton decomposition SD a standard tableau?

When is SD(w) standard?

Example

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \mathrm{SD}(21435) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \qquad \mathrm{SD}(31425) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Theorem (UConn 2020 + D. Grinberg)

Given a permutation w, the following are equivalent:

1. $\mathrm{SD}(w)$ is standard
2. $\mathrm{SD}(w) = \mathrm{P}(w)$
3. the shape of $\mathrm{SD}(w)$ is equal to the shape of $\mathrm{P}(w)$

Definition

We say that a permutation w is *BBS-good* (or *good* for short) if the tableau $\mathrm{SD}(w)$ is standard.

Q(w) determines whether w is good

Proposition

Given a Q-equivalence class, either all permutations in it are good or all of them are not good.

Definition (Good tableaux)

A standard tableau T is *good* if each permutation whose Q tableau equals T is good.

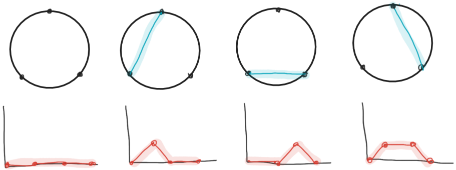
- How many good tableaux are there?

Good tableaux and Motzkin numbers

Conjecture (partial result, SUMRY 2022)

The good standard tableaux, $\{\mathrm{Q}(w) \mid w \in S_n \text{ and } \mathrm{SD}(w) \text{ is standard}\}$, are counted by the Motzkin numbers:

$$M_0 = 1, \qquad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}$$



$n = 3$

The first few Motzkin numbers are 1, 1, 2, 4, 9, 21, 51, 127, 323, 835.

Knuth Relations

Suppose v, $w \in S_n$ and $x < y < z$.

1. v and w differ by a Knuth relation of the **first kind** (K_1) if
$$v = x_1 \dots yxz \dots x_n \text{ and } w = x_1 \dots yzx \dots x_n \text{ or vice versa}$$
2. v and w differ by a Knuth relation of the **second kind** (K_2) if
$$v = x_1 \dots xzy \dots x_n \text{ and } w = x_1 \dots zxy \dots x_n \text{ or vice versa}$$

In addition, v and w differ by a Knuth relation of **both kinds** (K_B) if they differ by K_1 and they differ by K_2, that is,

$$v = x_1 \dots y_1xzy_2 \dots x_n \text{ and } w = x_1 \dots y_1zxy_2 \dots x_n \text{ or vice versa}$$

where $x < y_1, y_2 < z$

Example $\mathbf{326}154 \sim^{K_1} \mathbf{362}154$ $362\mathbf{154} \sim^{K_B} 362\mathbf{514}$

We say that v and w are *Knuth equivalent* if they differ by a finite sequence of Knuth relations.

P-tableaux and Knuth moves

Theorem (Knuth, 1970)

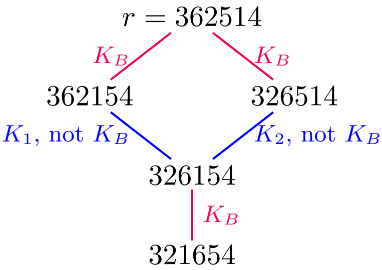
- *There is a path of Knuth moves from w to the row reading word of $P(w)$.*
- *Two permutations have the same P tableau iff they are in the same Knuth equivalence class.*

Example

The Knuth equivalence class of the row reading word $r = 362514$ of

1	4
2	5
3	6

:



Soliton decompositions and Knuth moves

The soliton decomposition is preserved by non-K_B Knuth moves, but one K_B move changes the soliton decomposition.

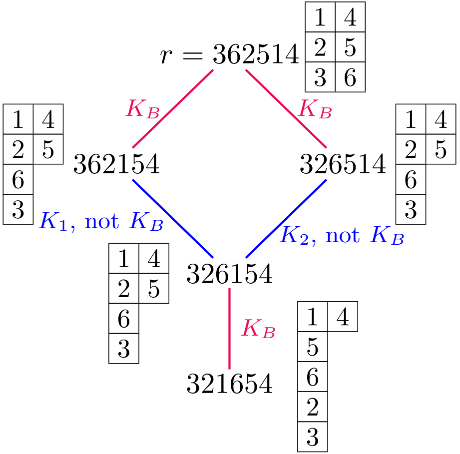
Theorem (UConn Math REU 2020)

Let r denote the row reading word of $\mathrm{P}(w)$.

- If there exists a path of *non-*K_B Knuth moves from w to r, then $\mathrm{SD}(w) = \mathrm{P}(w)$. In particular, $\mathrm{SD}(r) = \mathrm{P}(r)$.
- If there exists a path from w to r containing an *odd* number of K_B moves, then $\mathrm{SD}(w) \neq \mathrm{P}(w)$.

Example

Soliton decompositions of the Knuth equivalence class of 362154:



Thank you!

Y	O	U	!
A	N	K	
T	H		

Greene's theorem, slide 1/3

Definition (longest k-increasing subsequences)

A subsequence σ of w is called *k-increasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an increasing subsequence of w. Let $\mathrm{i}_k := \mathrm{i}_k(w)$ denote the length of a longest k-increasing subsequence of w.

Example (Let $w = 5623714$.)

- The longest 1-increasing subsequences are 567, 237, and 234.
- The longest 2-increasing subsequence is given by $562374 = 567 \sqcup 234$.
- A longest 3-increasing subsequence (among others) is given by $5623714 = 56 \sqcup 237 \sqcup 14$.
- Thus,

$$\mathrm{i}_1 = 3, \qquad \mathrm{i}_2 = 6, \qquad \text{and} \qquad \mathrm{i}_k = 7 \;\text{ if }\; k \geq 3.$$

Greene's theorem, slide 2/3

Definition (longest k-decreasing subsequences)

Similarly, a subsequence σ of w is called *k-decreasing* if, as a set, it can be written as a disjoint union

$$\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_k$$

where each σ_i is an decreasing subsequence of w. Let $\mathrm{d}_k := \mathrm{d}_k(w)$ denote the length of a longest k-decreasing subsequence of w.

Example (Let $w = 5623714$.)

- The longest 1-decreasing subsequences are 521, 621, 531, and 631.
- A longest 2-decreasing subsequence (among others) is given by $52714 = 521 \sqcup 74$.
- A longest 3-decreasing subsequence (among others) is given by $5623714 = 52 \sqcup 631 \sqcup 74$.
- Thus,

$$\mathrm{d}_1 = 3, \qquad \mathrm{d}_2 = 5, \qquad \text{and} \qquad \mathrm{d}_k = 7 \text{ if } k \geq 3.$$

Greene's theorem, slide 3/3

Theorem (Greene, 1974)

Suppose $w \in S_n$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ denote the RS partition of w, that is, let $\lambda = \mathrm{sh}\, P(w)$. Let $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ denote the conjugate of λ. Then, for any k,

$$\mathrm{i}_k(w) = \lambda_1 + \lambda_2 + \ldots + \lambda_k,$$
$$\mathrm{d}_k(w) = \mu_1 + \mu_2 + \ldots + \mu_k.$$

Example

By Greene's theorem, the RS partition is equal to $\lambda = (\mathrm{i}_1, \mathrm{i}_2 - \mathrm{i}_1, \mathrm{i}_3 - \mathrm{i}_2) = (3, 3, 1)$. We can verify this by computing the RS tableaux

$$P(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 \\ \cline{1-1} \end{array}, \qquad Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}.$$

A localized version of Greene's theorem, slide 1/3

Definition (A localized version of longest k-increasing subsequences)

Let $\mathrm{i}(u) :=$ the length of a longest increasing subsequence of u.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{I}_k(w) = \max_{w=u_1|\cdots|u_k} \sum_{j=1}^{k} \mathrm{i}(u_j)$, where the maximum is taken over ways of writing w as a concatenation $u_1 \mid \cdots \mid u_k$ of consecutive subsequences.

Example

Let $w = 5623714$. For short, we write $\mathrm{I}_k := \mathrm{I}_k(w)$. Then

$\mathrm{I}_1 = \mathrm{i}(w) = 3$ (since the longest increasing subsequences are 567, 237, and 234),

$\mathrm{I}_2 = 5$ (witnessed by $56|23714$ or $56237|14$),

$\mathrm{I}_3 = 7$ (witnessed uniquely by $56|237|14$), and

$\mathrm{I}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem, slide 2/3

Definition (A localized version of longest k-decreasing subsequences)

Let $\mathrm{D}(u) \coloneqq 1 + |\{\text{descents of } u\}|$.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{D}_k(w) = \max_{w = u_1 \sqcup \cdots \sqcup u_k} \sum_{j=1}^{k} \mathrm{D}(u_j)$, where the maximum is taken over ways to write w as the union of disjoint subsequences u_j of w.

Example

Let $w = 5623714$. For short, we write $\mathrm{D}_k \coloneqq \mathrm{D}_k(w)$. Then

$\mathrm{D}_1 = \mathrm{D}(w) = 1 + |\text{descents of } 5623714| = 1 + |\{2, 5\}| = 3$,
$\mathrm{D}_2 = 6$ (one can take subsequences 531 and 6274, among other partitions),
$\mathrm{D}_3 = 7$ (one can take subsequences 52, 631, and 74, among other partitions), and
$\mathrm{D}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem, slide 3/3

Theorem (Lewis–Lyu–Pylyavskyy–Sen 2019)

Suppose $w \in S_n$. *Let* $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \ldots)$ *denote* $\operatorname{sh}\operatorname{SD}(w)$. *Let* $M = (M_1, M_2, M_3, \ldots)$ *denote the conjugate of* Λ. *Then, for any* k,

$$\mathrm{I}_k(w) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_k,$$
$$\mathrm{D}_k(w) = M_1 + M_2 + \ldots + M_k.$$

Example

Let $w = 5623714$. By the above theorem, $\operatorname{sh}\operatorname{SD}(w) = (\mathrm{I}_1, \mathrm{I}_2 - \mathrm{I}_1, \mathrm{I}_3 - \mathrm{I}_2) = (3, 2, 2)$. We can verify this by computing the soliton decomposition $\operatorname{SD}(w)$, which turns out to be the (non-standard) tableau

1	3	4
2	7	
5	6	

.

Note: $\operatorname{sh}\operatorname{SD}(w) = (3, 2, 2)$ is smaller than $\operatorname{sh} P(w) = (3, 3, 1)$ in the dominance order.

Examples: permutations with L-shaped SD

A permutation with L-shaped SD which is not a column reading word:

$w = 3217654 = (13)(47)(56)$ is a noncrossing involution.

$$\mathrm{P}(w) = \mathrm{Q}(w) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 & \\ \cline{1-1} \end{array} \qquad \text{and} \qquad \mathrm{SD}(w) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & \\ \cline{1-1} 6 & \\ \cline{1-1} 7 & \\ \cline{1-1} 2 & \\ \cline{1-1} 3 & \\ \cline{1-1} \end{array}$$

An involution which is neither noncrossing nor a column reading word:

$v = 5274163 = (15)(37)$ has a crossing.

$$\mathrm{P}(v) = \mathrm{Q}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 5 & 7 & \\ \cline{1-2} \end{array} \qquad \text{and} \qquad \mathrm{SD}(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 4 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} 7 & & \\ \cline{1-1} 5 & & \\ \cline{1-1} \end{array}$$

Permutations connected by K_B moves and have the same SD

Two permutations with the same SD which are connected by K_B moves:

$r = 35124 \qquad \mathrm{SD}(r) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$

K_B

$31524 \qquad \mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$

K_2, not K_1

$31254 \qquad \mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$

K_2, not K_1

$13254 \qquad \mathrm{SD} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$

K_B

$w = 13524 \qquad \mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$