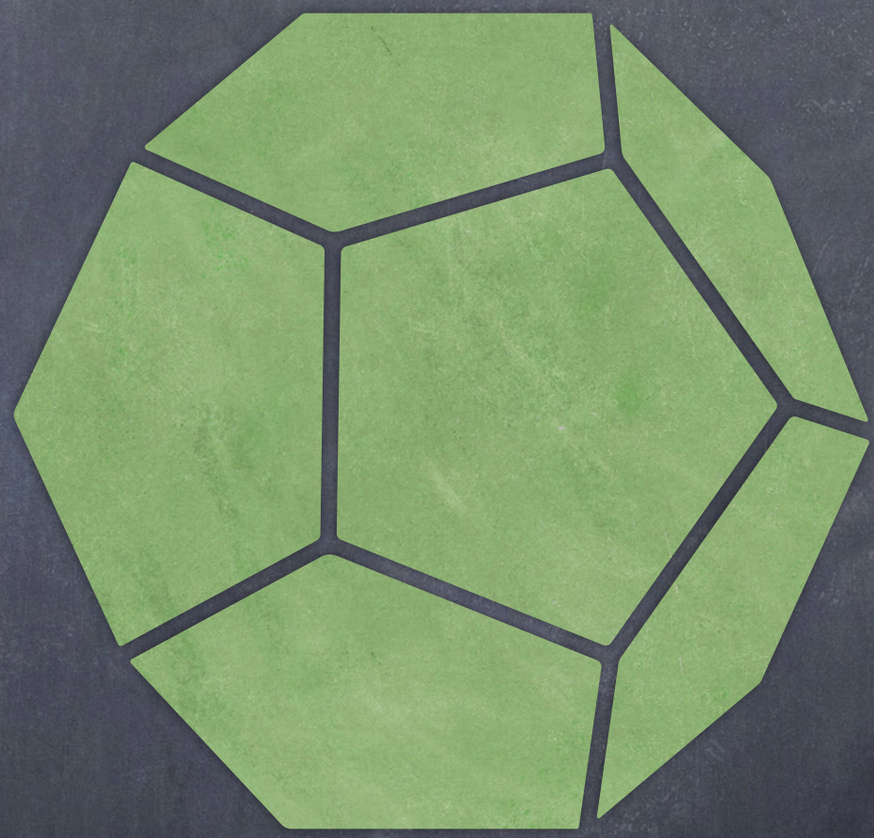
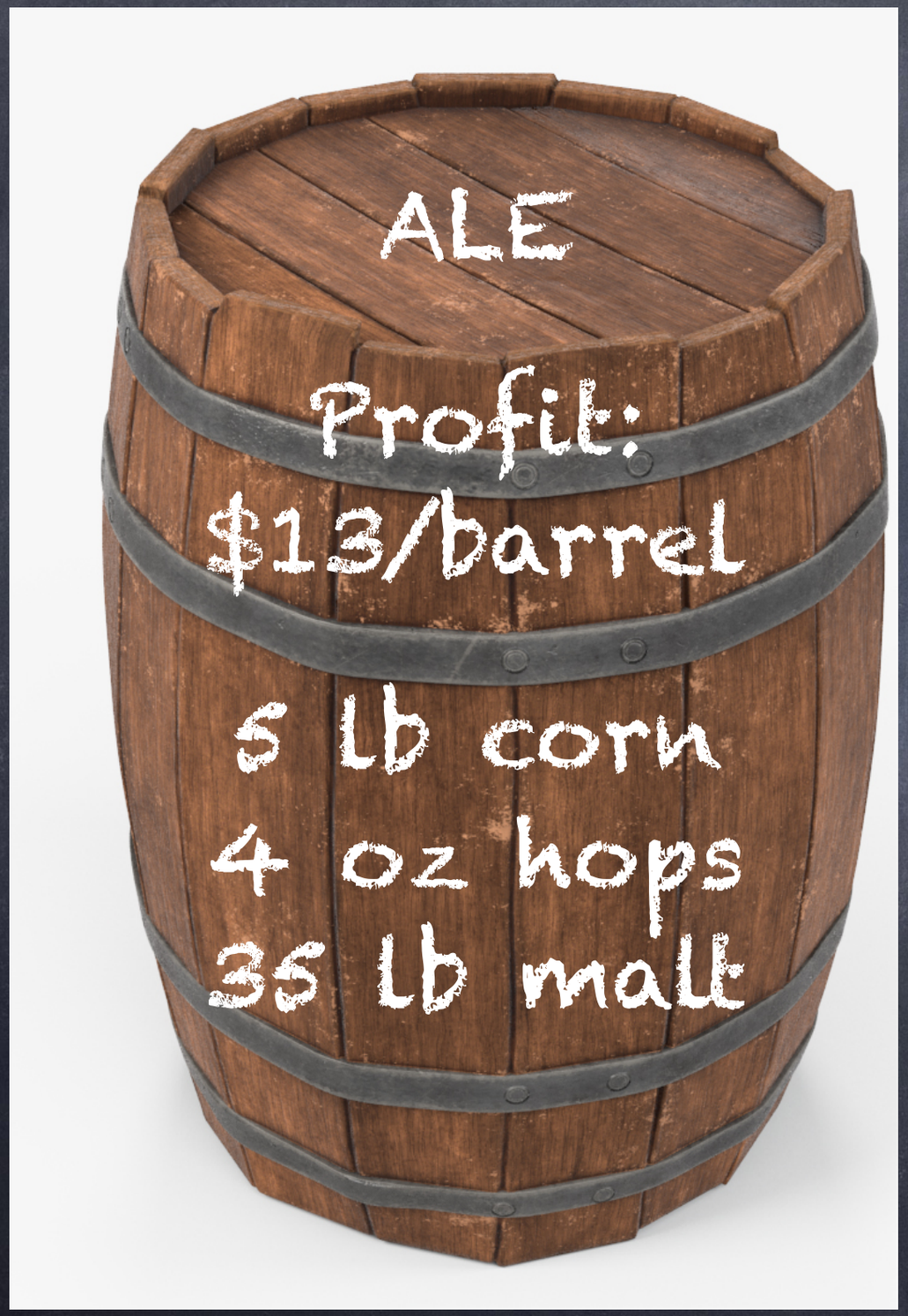
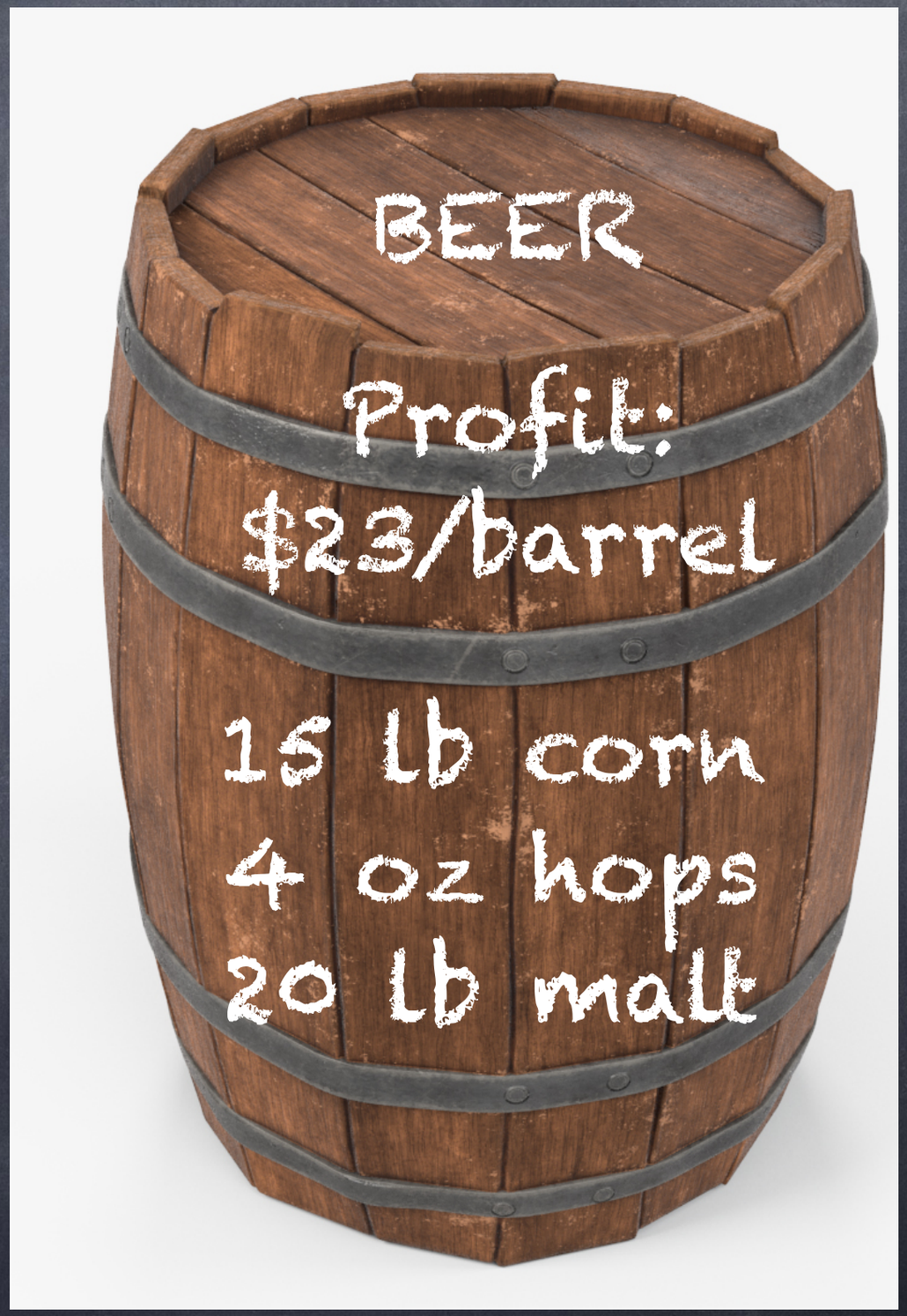
Polytopes and the brewery problem



A brewery produces ale and beer



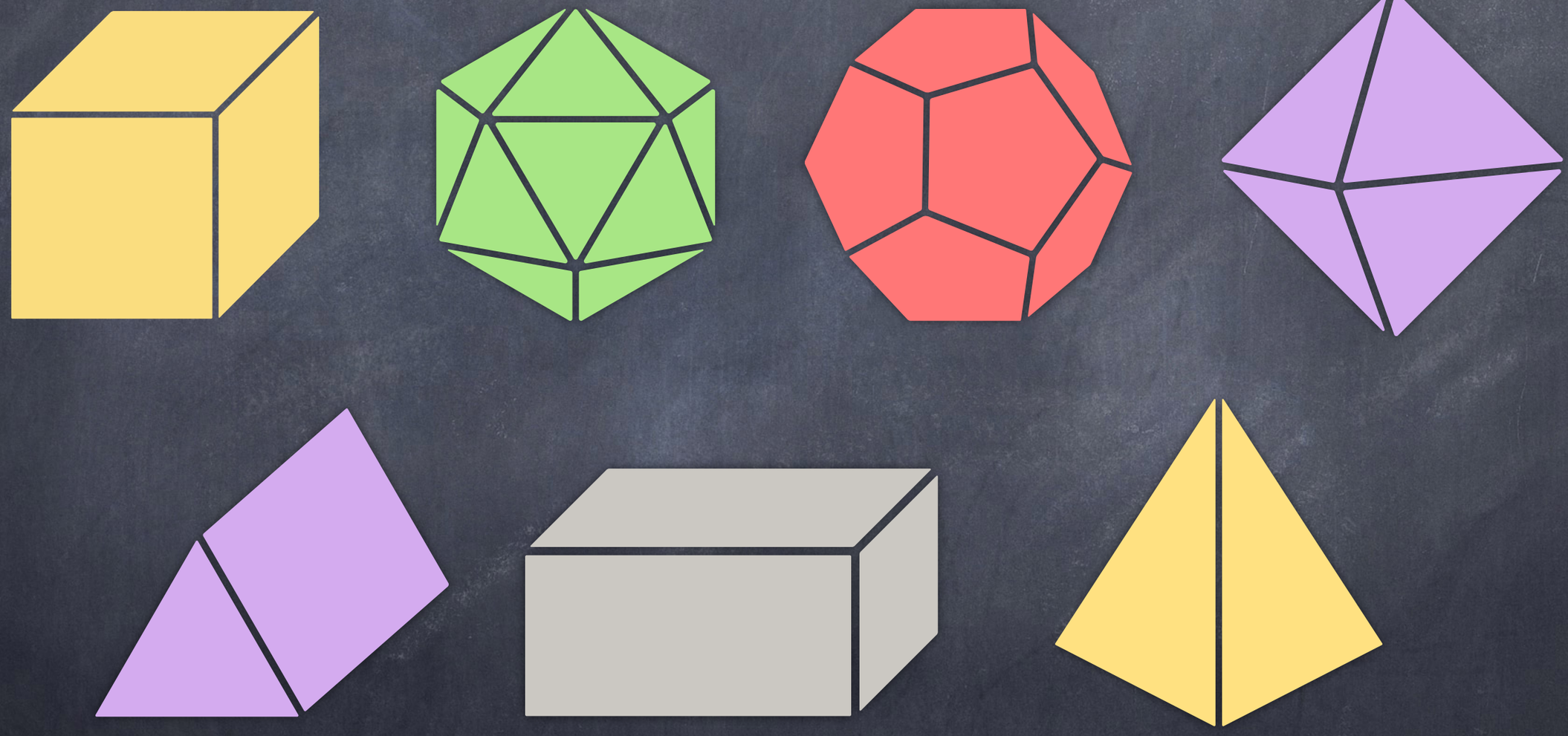


- Limited resources: The brewery only has 480 lb of corn, 160 oz of hops, and 1190 lb of malt.
- How many barrels of ale and beer should be produced to maximize total profit given the existing resources?

What is ...

a convex polytope?

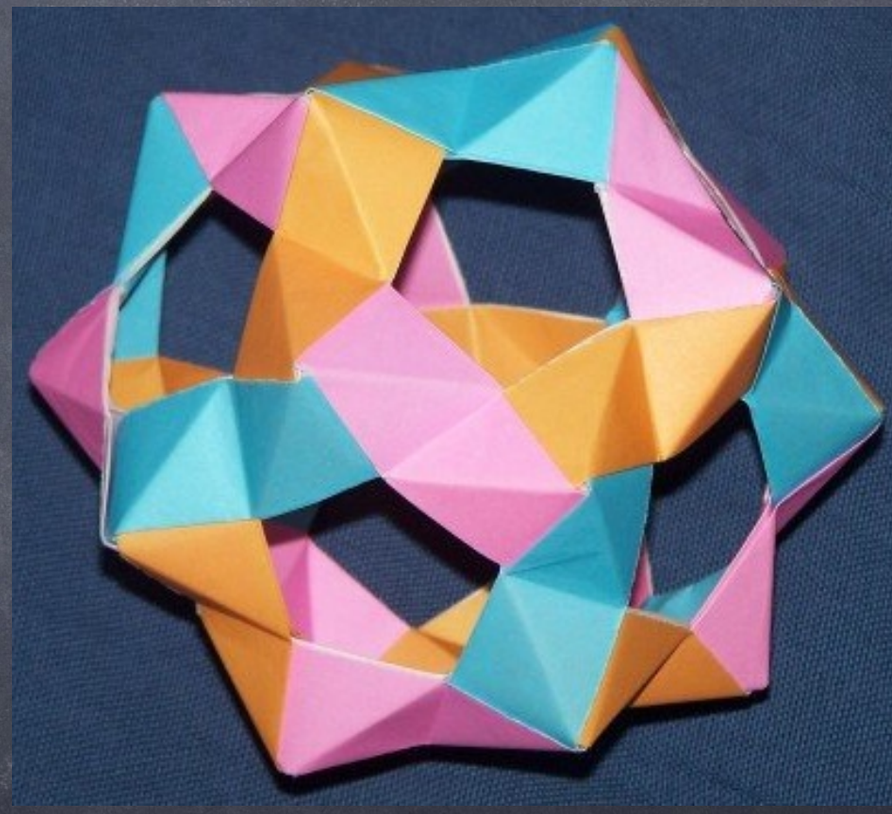
Convex polytopes



Convex polytopes

The five platonic solids

Not convex

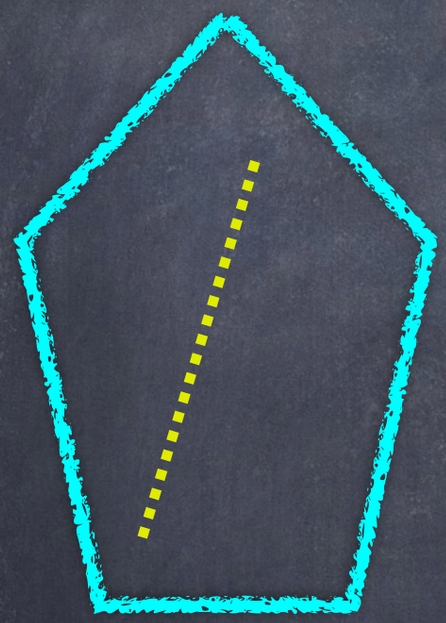


Origami by Madeline Handschy

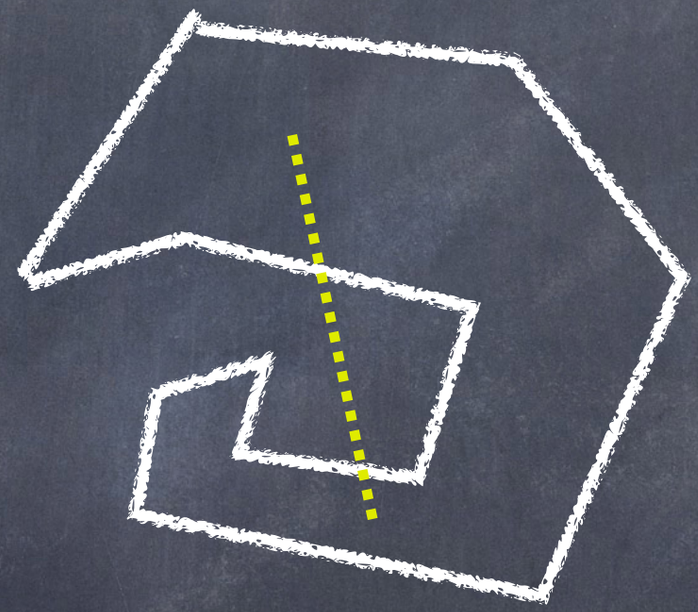
Not convex

Stamps featuring non-convex polytopes

DEFINITION: A region is convex if it contains every line segment joining two of its points



Convex

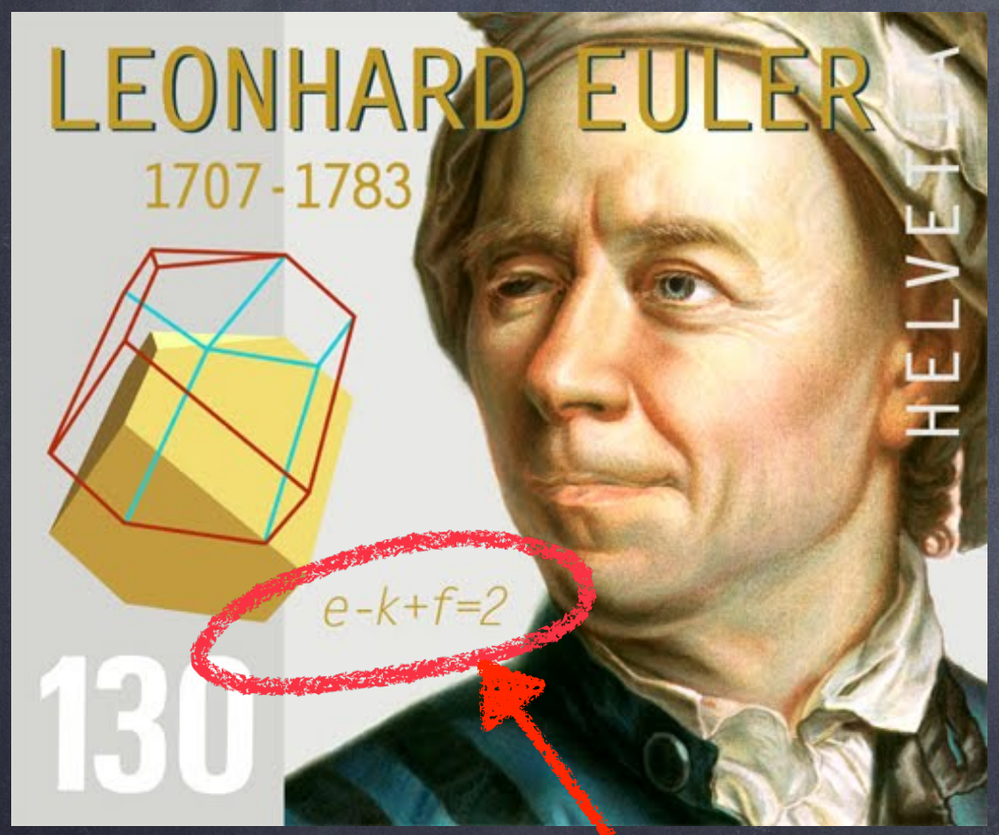


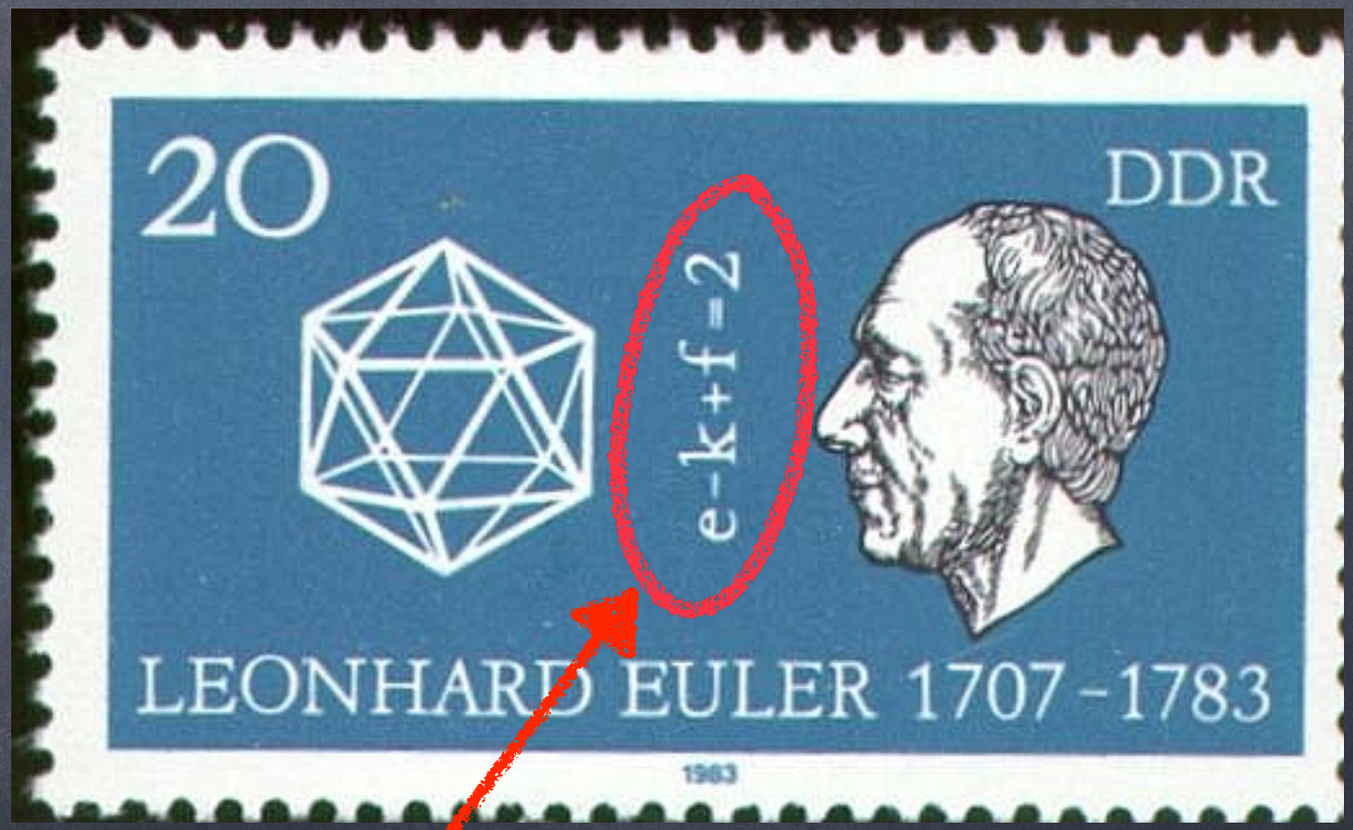
Not convex

Leonhard Euler (1707-1783)

Switzerland 1957, 2007; East Germany 1950, 1983; Russia 1957; Korea 2014

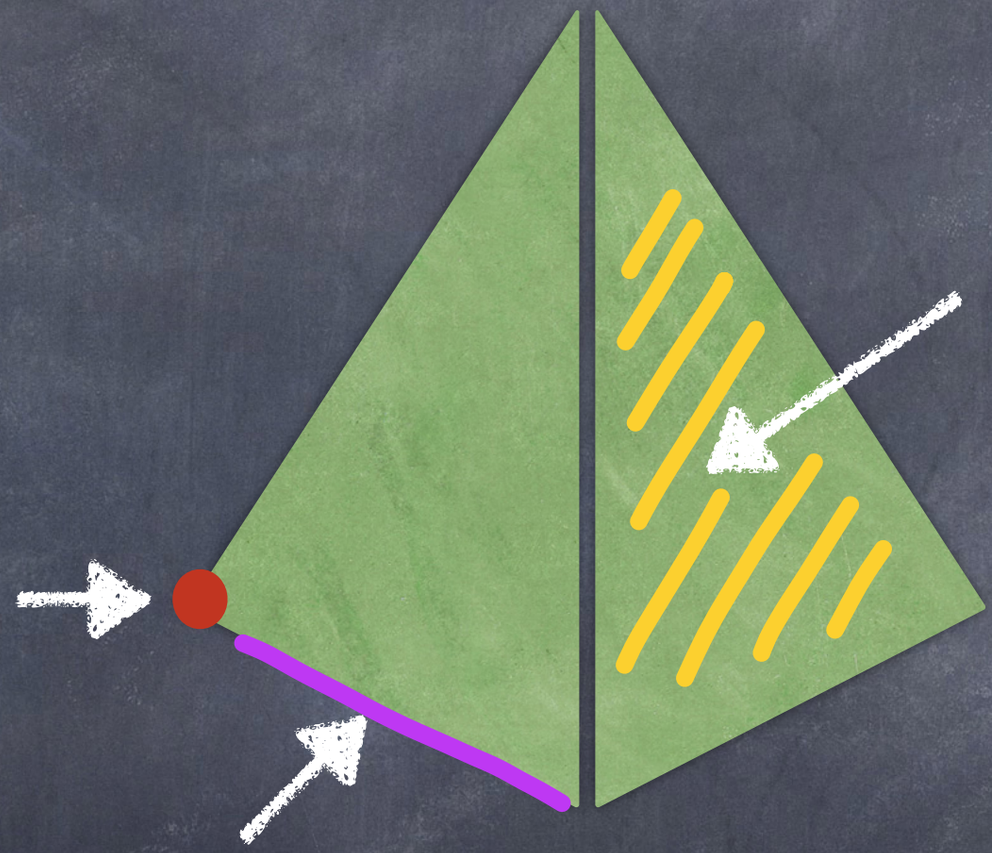
Counting faces of a convex polytope





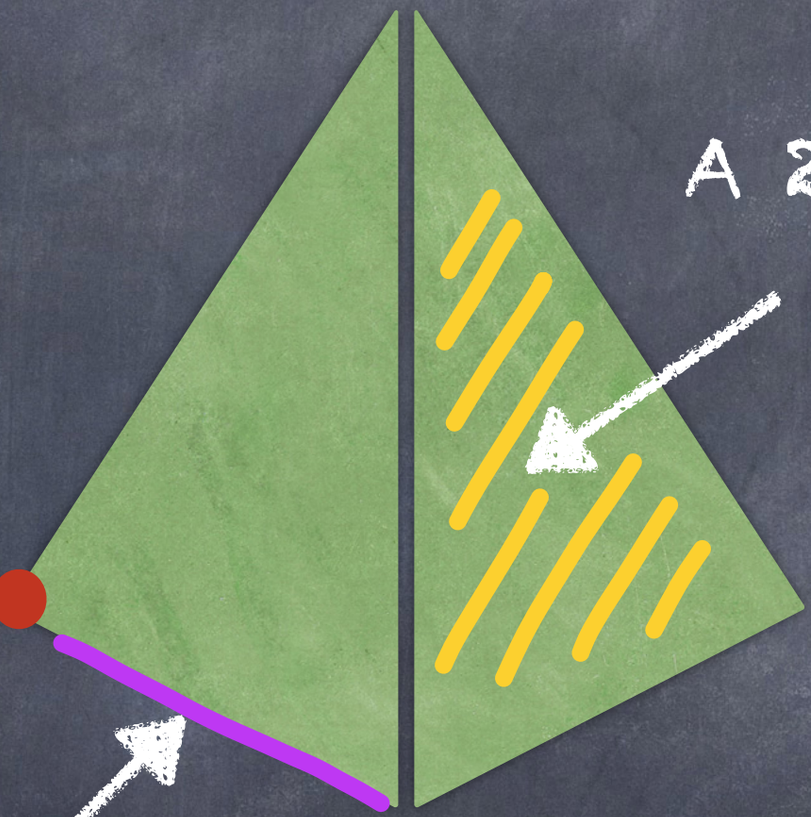
What are these?!

What are ... the faces of a polytope?



A tetrahedron

k-th dimensional faces



A 2-dimensional face

A 0-dimensional face (vertex)

A 1-dimensional face (edge)

A 3-dimensional face (the entire tetrahedron)

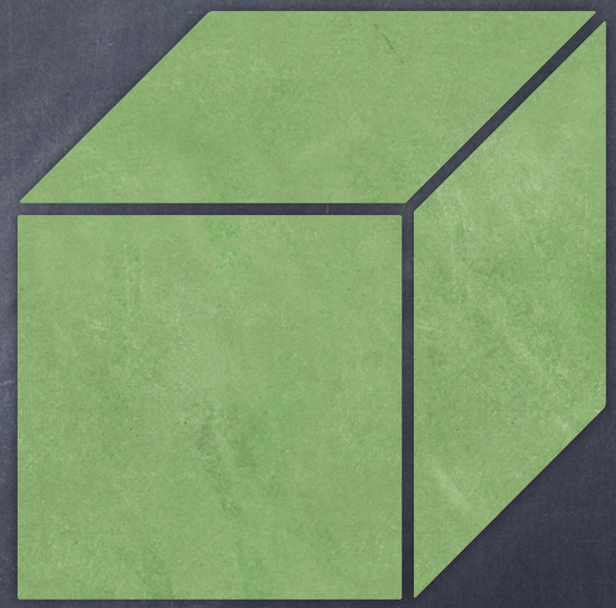
Counting faces

The face-vector or f-vector of a 3D polytope is (f0, f1, f2), where

f0 = # of the 0-dimensional faces (vertices)

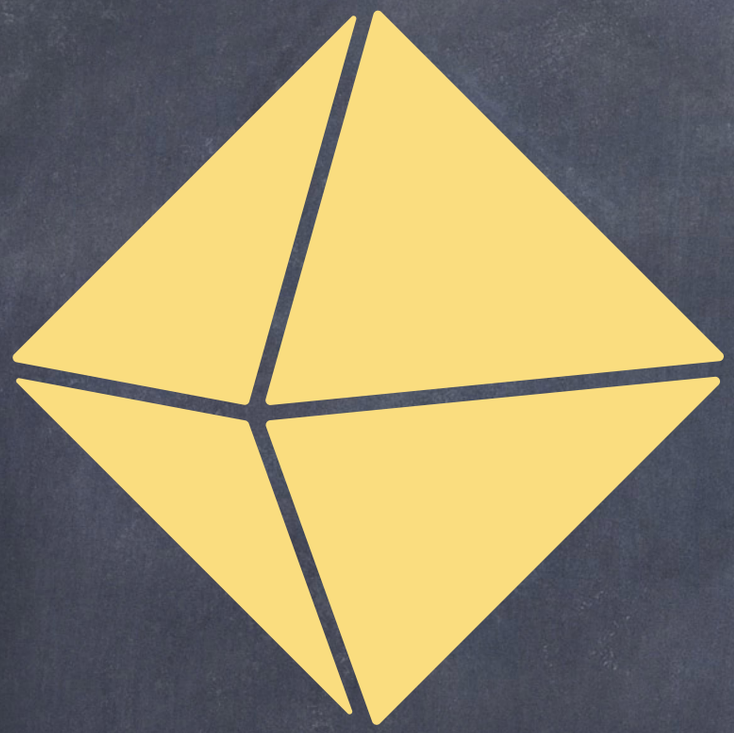
f1 = # of the 1-dimensional faces (edges)

f2 = # of the 2-dimensional faces



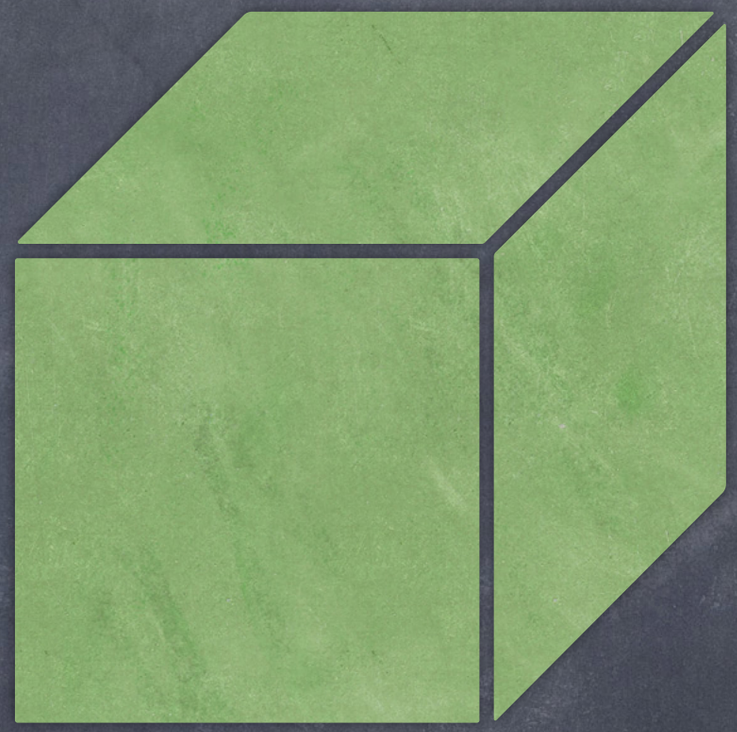
$(f_0, f_1, f_2) = (8,12,6)$

(f0, f1, f2) ?



Octa(8)hedron

(6,12,8)

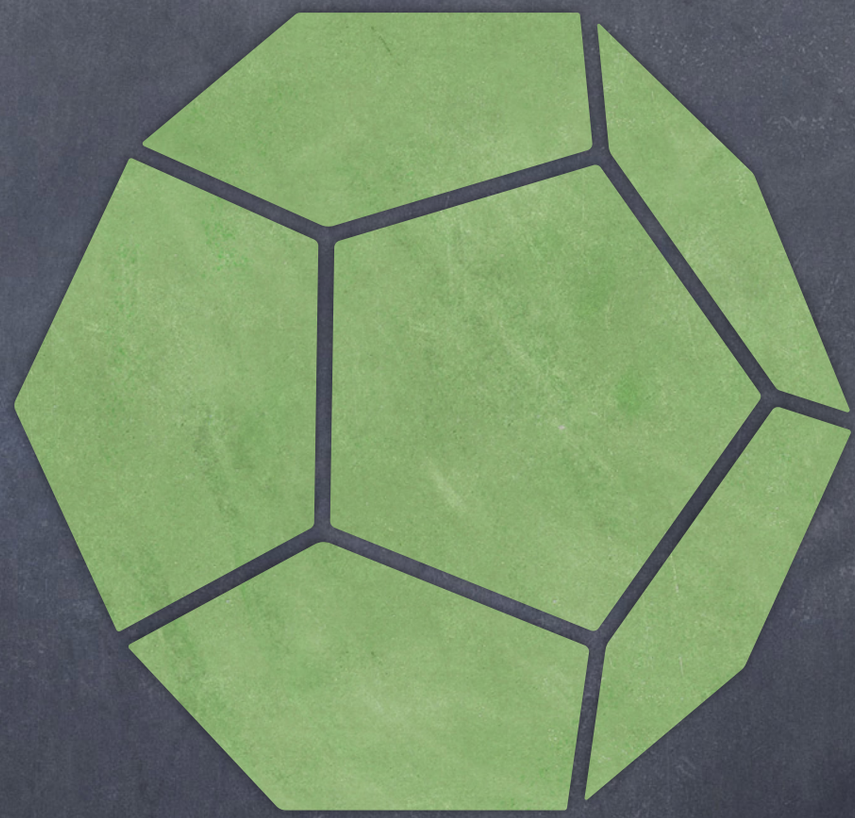


Hexa(6)hedron

(8,12,6)

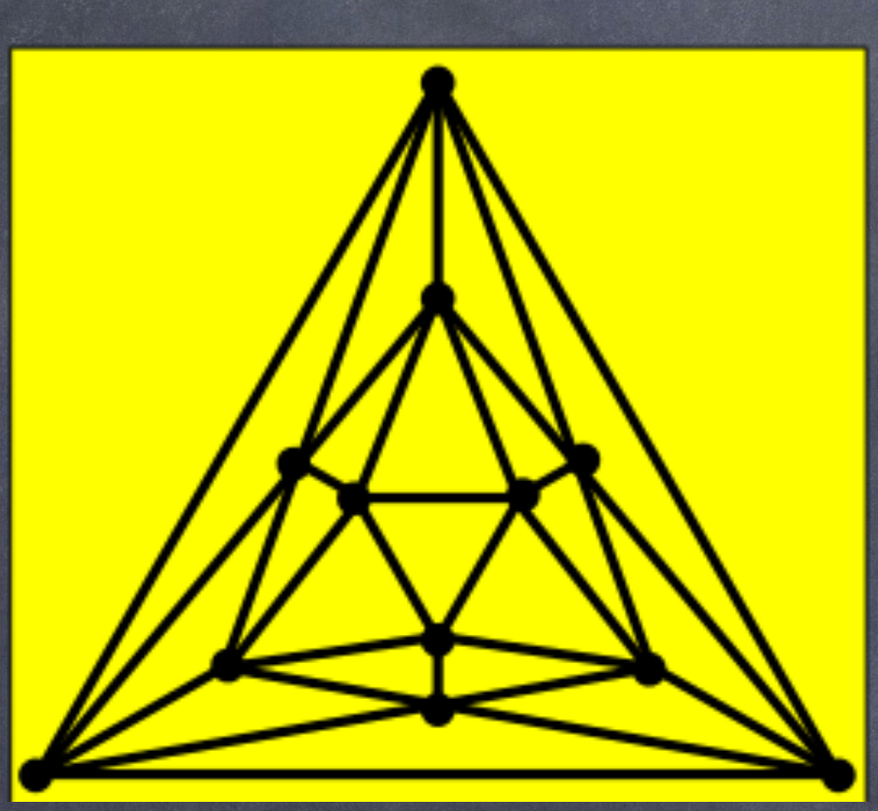
(f_0, f_1, f_2) ?

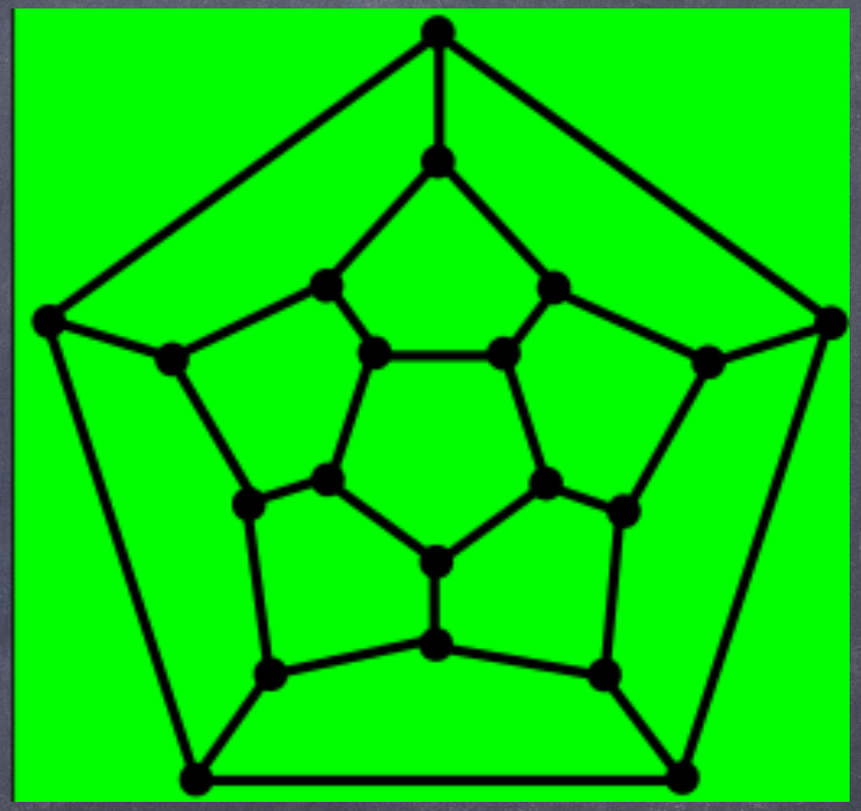
Icosa(20)hedron



Dodeca(12)hedron

(f0, f1, f2) ?





from Wikimedia Commons

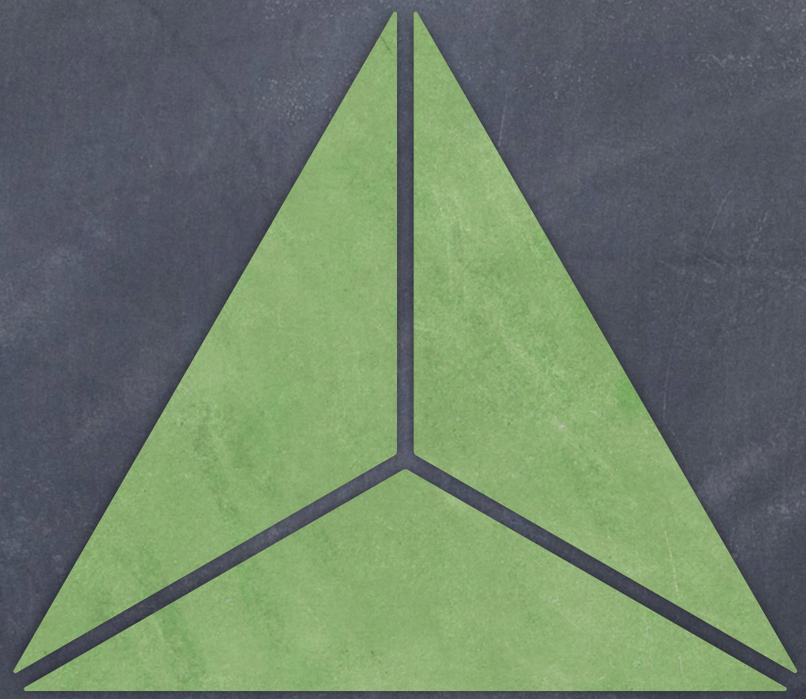
Icosa(20)hedron (12,30,20)

Dodeca(12)hedron (20,30,12)

(f0, f1, f2) ?

Tetra(4)hedron

(4,6,4)

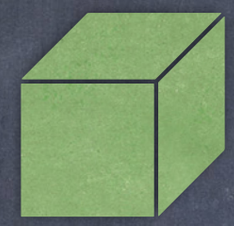


Tetra(4)hedron

(4,6,4)

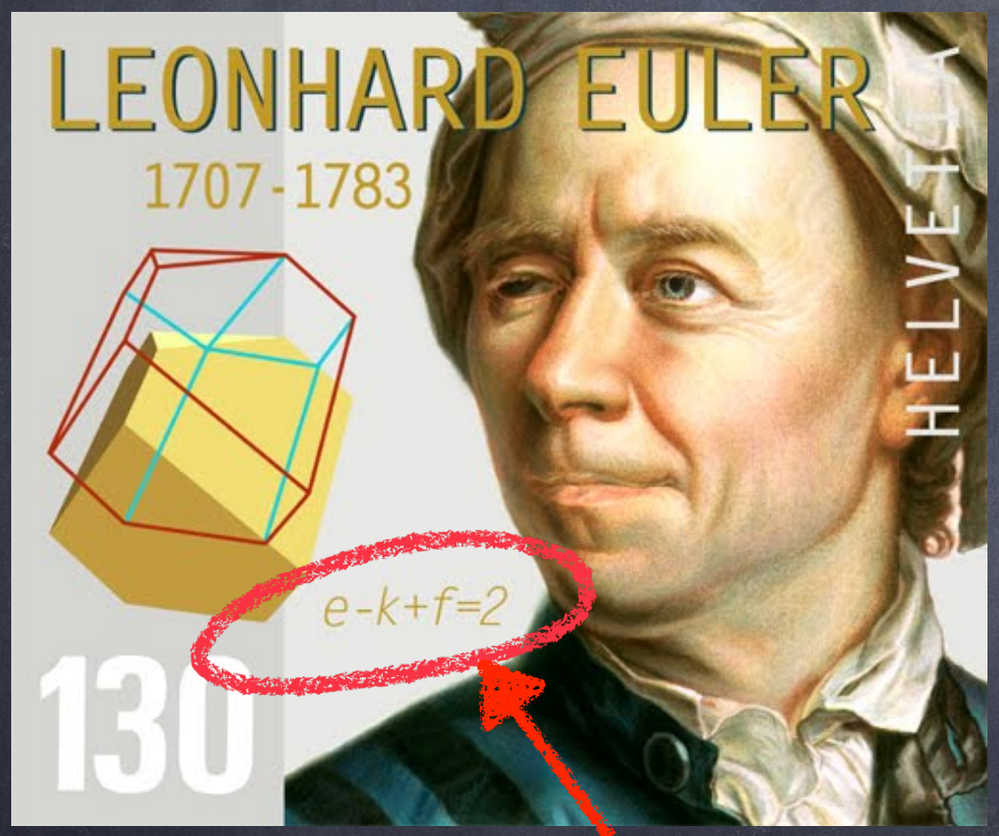
Euler's formula $f_0 - f_1 + f_2 = \ldots$

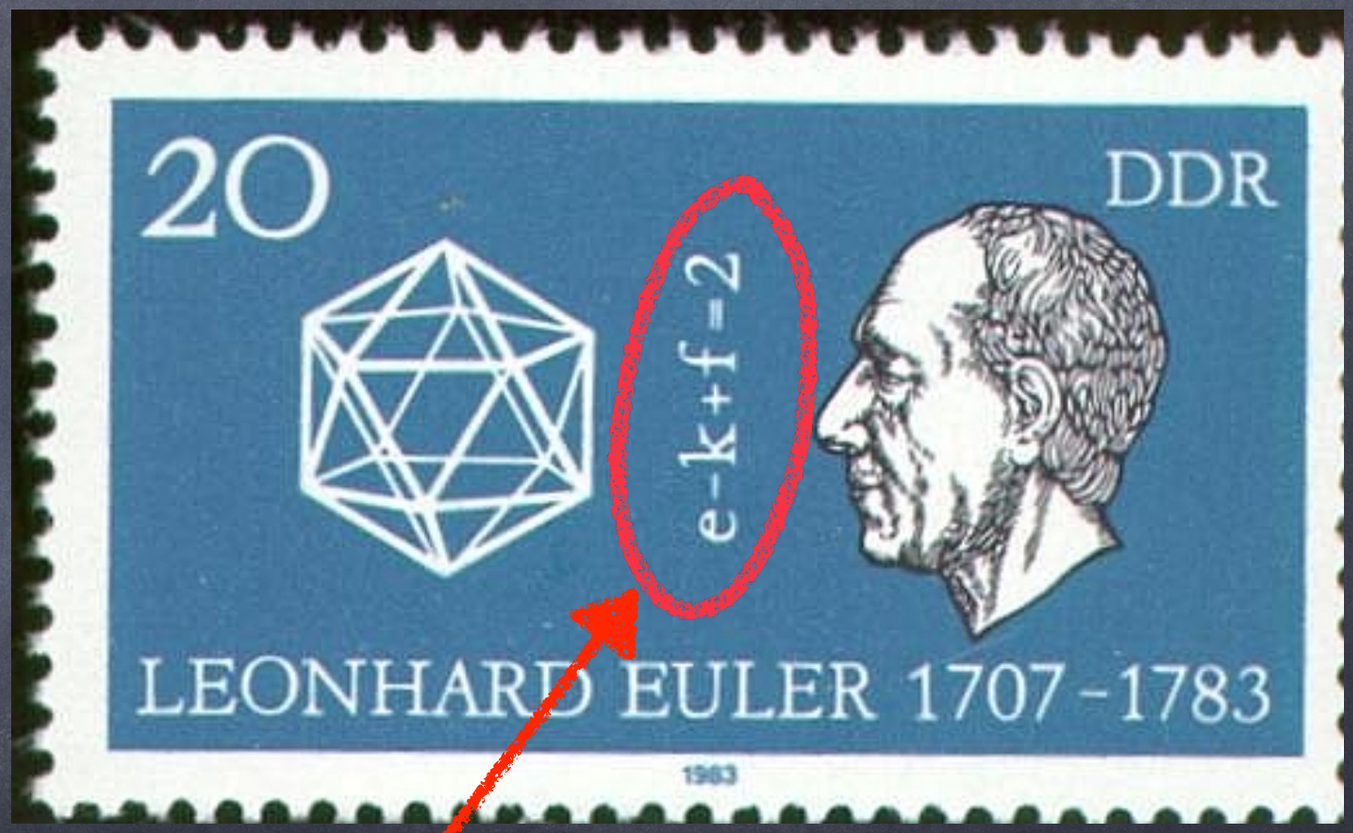
- (f0, f1, f2)
- Cube: (8, 12, 6)
- Octahedron: (6, 12, 8)
- Dodecahedron: (20, 30, 12)
- Icosahedron: (12, 30, 20)
- Tetrahedron: (4, 6, 4)





Euler's formula $f_0 - f_1 + f_2 = \ldots$





Euler's formula in German

What is ... a half-space?

- Points (x,y) where

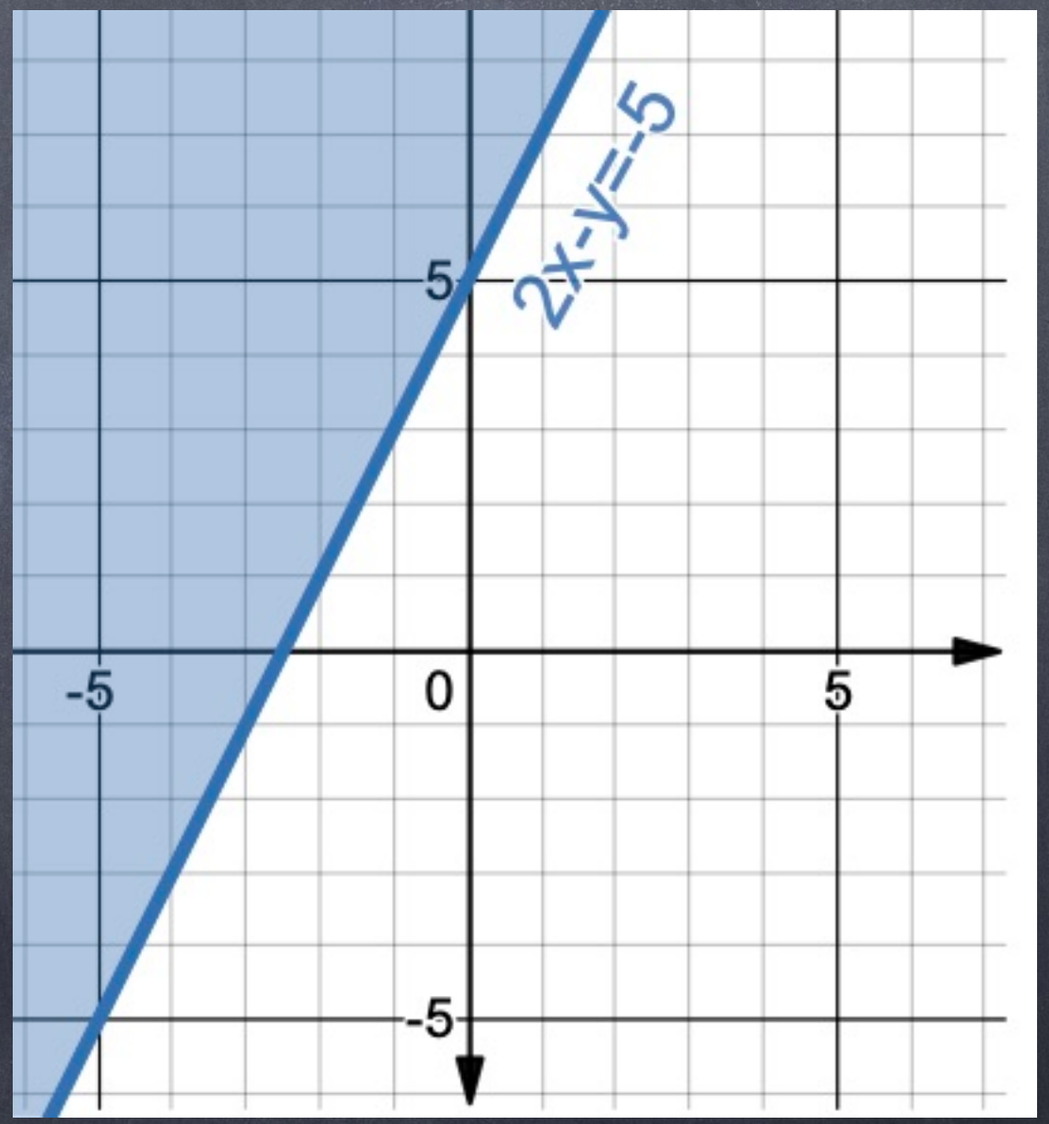
$$2x - y = -5$$

is a line

- Points (x,y) where

$$2x - y \leq -5$$

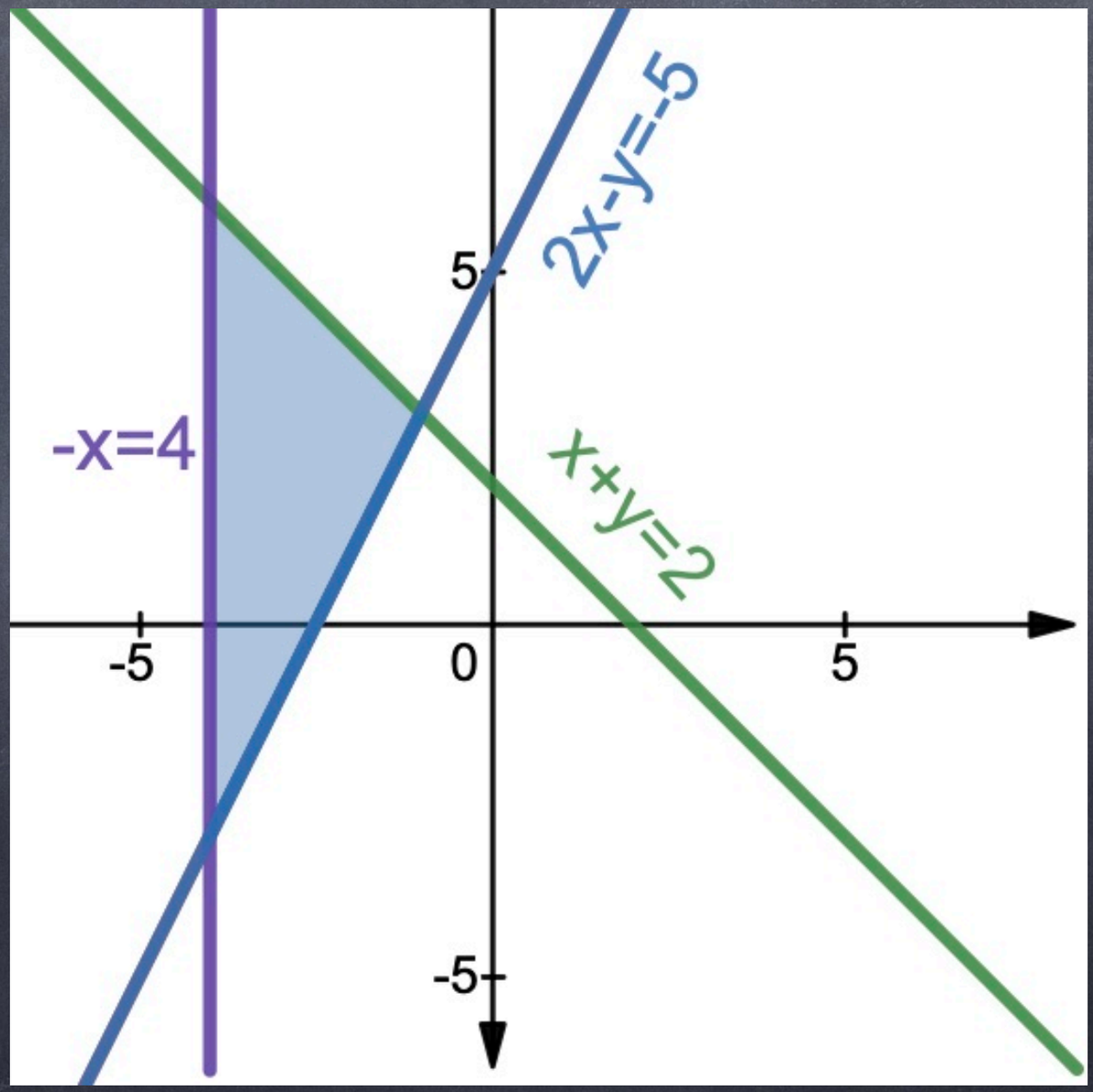
is a half-space



DEFINITION: A convex polytope is the intersection of finitely many half-spaces

Points (x,y) satisfying a system of inequalities:

$$2x - y \leq -5$$
$$x + y \leq 2$$
$$-x \leq 4$$



What is ... a half-space?

- Points (x,y,z) where

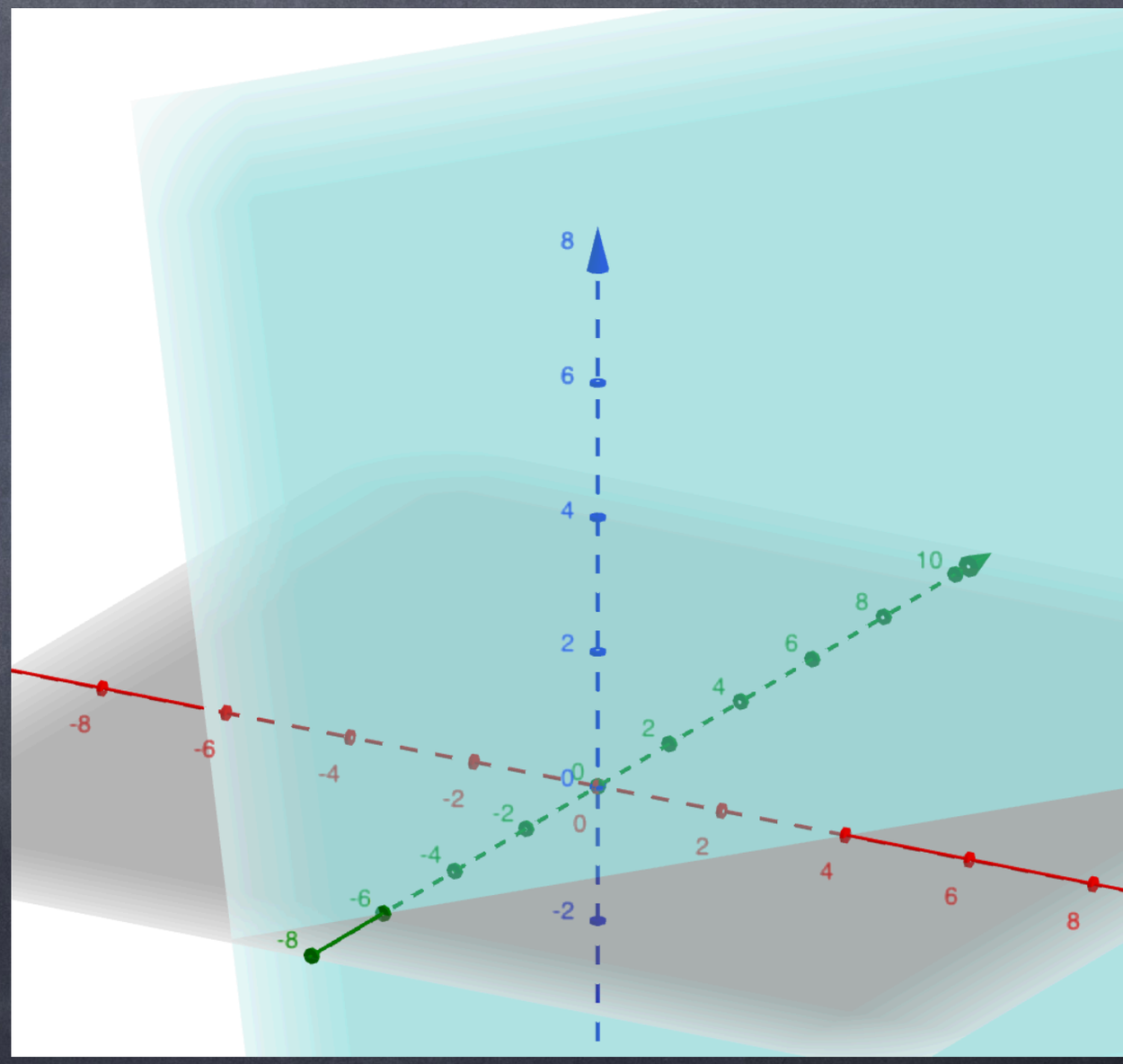
$$3x - 2y + z = 12$$

is a plane

- Points (x,y,z) where

$$3x - 2y + z \leq 12$$

is a half-space



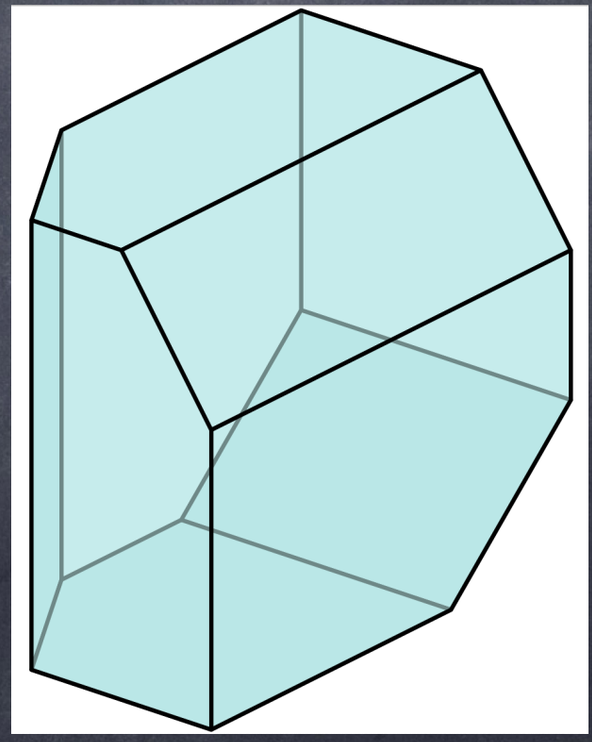
- Solutions to linear equation

$a_1x_1 + a_2x_2 + a_3x_3 = b$ is a plane

- Solutions to linear inequality

$a_1x_1 + a_2x_2 + a_3x_3 \leq b$ is a half-space

- Solutions to a system of linear inequalities is ...



from Wikimedia Commons

DEFINITION: A convex polytope is the intersection of finitely many half-spaces

A half-space is
A linear inequality

$$a_1x_1 + a_2x_2 + \ldots + a_dx_d \leq b$$

A convex polytope
is solutions to
a system of
linear inequalities

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,d}x_d &\leq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,d}x_d &\leq b_2 \\
&\vdots \\
a_{k,1}x_1 + a_{k,2}x_2 + \ldots + a_{k,d}x_d &\leq b_d
\end{aligned}$$

- A brewery produces ale and beer
- Limited quantity of corn, hops, malt

Beverage	Ale	Beer	Quantity
Corn (lb)	5	15	480
Hops (oz)	4	4	160
Malt (lb)	35	20	1190
Profit/barrel ($)	13	23	

How can we maximize profits?

Devote all resources to ale: 1190/35=34 barrels of ale; profit: $442

Devote all resources to beer: 480/15=32 barrels of beer; profit: $736

12 barrels of ale, 28 barrels of beer; profit: $800

Beverage	Ale	Beer	Quantity
Corn (lb)	5	15	480
Hops (oz)	4	4	160
Malt (lb)	35	20	1190
Profit/gallon ($)	13	23	

Let A = # of barrels of ale

Let B = # of barrels of beer

Corn (lb)
Hops (oz)
Malt (lb)

$$5A + 15B \leq 480$$

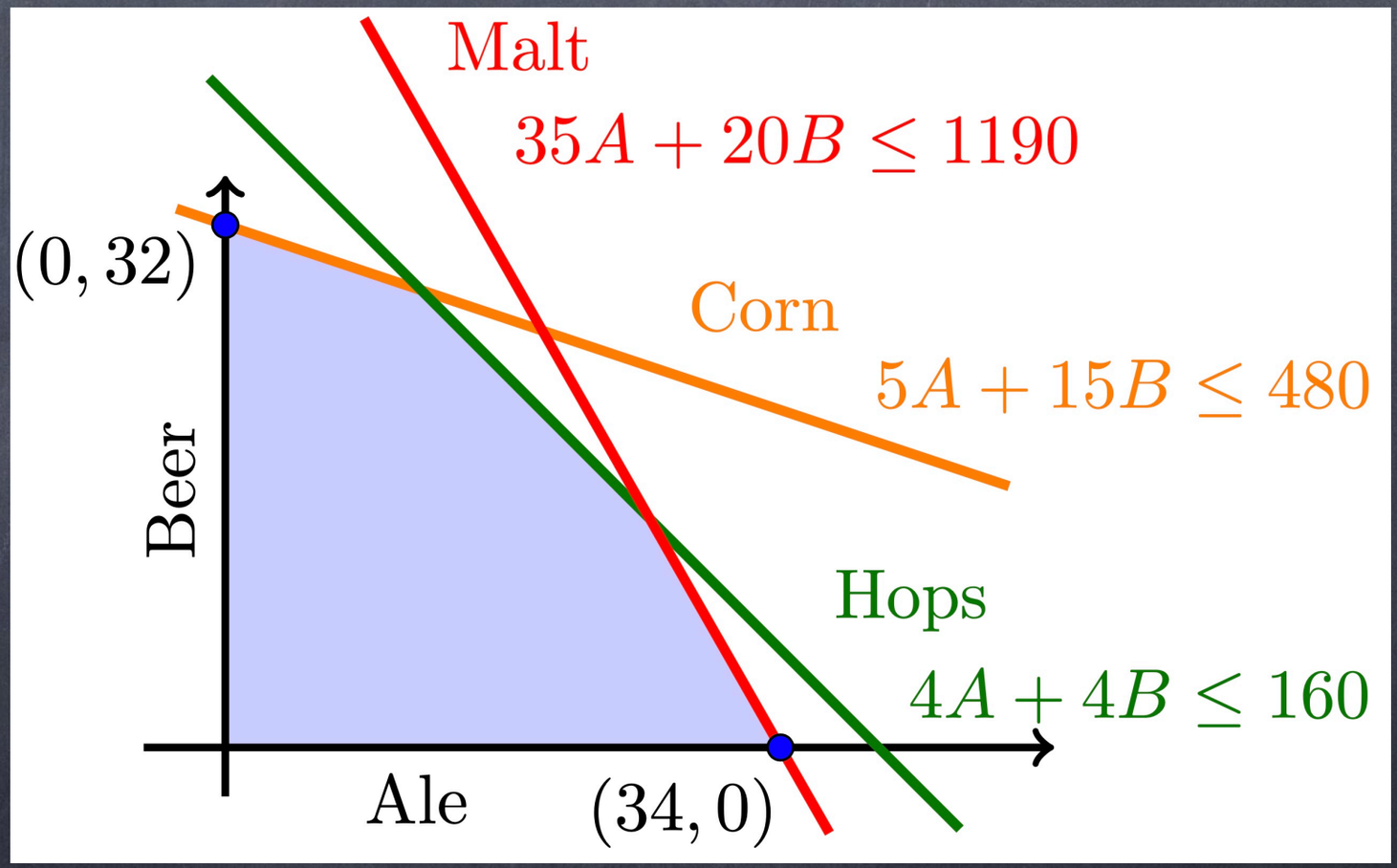
$$4A + 4B \leq 160$$

$$35A + 20B \leq 1190$$

$$0 \leq A,\ 0 \leq B$$

Restrictions due to limited resources

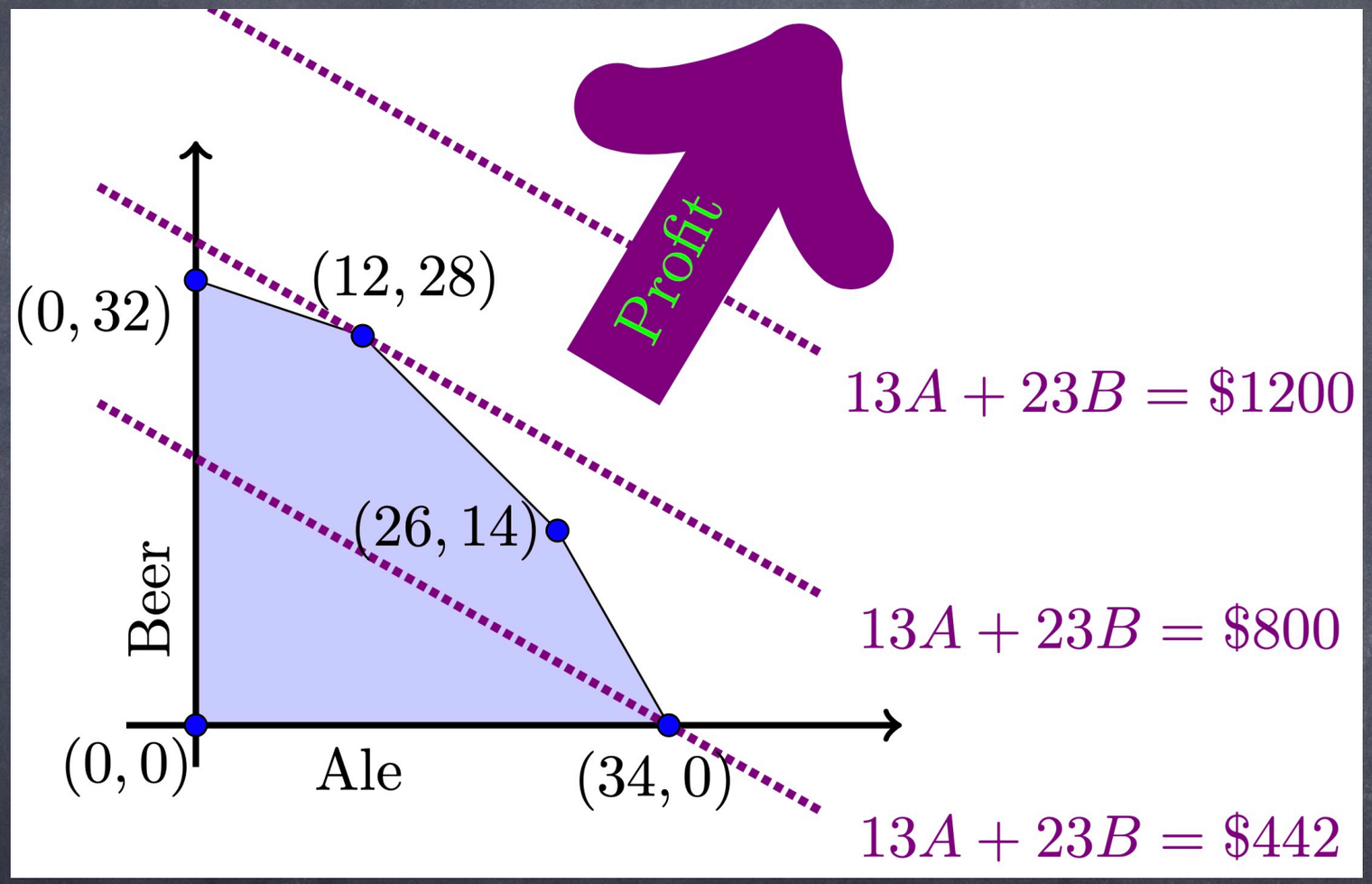
$$5A + 15B \leq 480$$
$$4A + 4B \leq 160$$
$$35A + 20B \leq 1190$$
$$0 \leq A,\ 0 \leq B$$



Beverage	Ale	Beer
Profit/barrel ($)	13	23

Total profit:

$$13A + 23B$$



Optimal solution occurs at an extreme point

Other applications

- Linear programming: airline crew assignment, data mining, radiation therapy, etc.
- NASA: Unfolding space telescopes for ease of transport
- Voting theory: Computing probability that different voting systems produce different winners
- Group theory: reflection groups, etc.
- Combinatorics: partial ordered sets, cluster algebras, etc.

THANK YOU!

Congratulations to the award winners!

This talk was inspired by an MAA Distinguished Lecture Series presentation by Prof. Jesús De Loera (UC Davis) in September 2012: "Easy to state but hard to solve: favorite open problems in polyhedral geometry"