Cambrian combinatorics on quiver representations (type A)

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Type A quiver representations

Q an orientation of A_n Dynkin diagram

$$s_1 - s_2 - \cdots - s_n$$

rep Q : objects = finite-dimensional representations of Q

morphisms = representation maps

Indecomposable representations are $M(i,j)$, $i \leq j$

cannot be written as a direct sum of two nonzero representations

$$0 - 0 \cdots - \underset{s_i}{\mathbb{C}} \overset{\mathrm{Id}}{-} \mathbb{C} \cdots \overset{\mathrm{Id}}{-} \underset{s_j}{\mathbb{C}} - 0 \cdots 0 - 0$$

$\overset{1:1}{\longleftrightarrow}$ positive roots of type A_n

Example

A_4

$$Q = \begin{tikzcd} s_1 \arrow[dr] & & s_3 \arrow[dl] & s_4 \arrow[l] \\ & s_2 & & \end{tikzcd}$$

$$M(1,3) = \begin{tikzcd} \mathbb{C} \arrow[dr, "\mathrm{Id}"] & & \mathbb{C} \arrow[dl, "\mathrm{Id}"'] & 0 \arrow[l, "0"'] \\ & \mathbb{C} & & \end{tikzcd}$$

Shorthand notation $\begin{matrix} 1 & & 3 \\ & 2 & \end{matrix}$

$\alpha_1 + \alpha_2 + \alpha_3$

Auslander – Reiten quiver

The Auslander – Reiten quiver Γ_Q of rep Q is a connected directed graph with

vertices = indecomposable representations — cannot be written as a direct sum of two nonzero representations

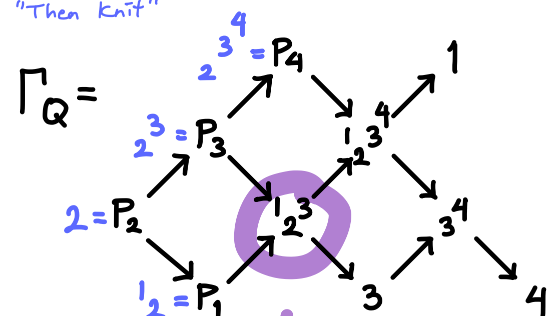
arrows = irreducible morphisms — does not factor through another representation

$$Q = \quad S_1 \to S_2 \leftarrow S_3 \leftarrow S_4$$

follow arrows to find P_4

Start with indecomposable projectives
"Then knit"

$\Gamma_Q =$



Caldero – Chapoton – Schiffler model 2006

— model for cluster category $\mathcal{D}^b(\text{rep } Q)/\tau^{-1}[1]$

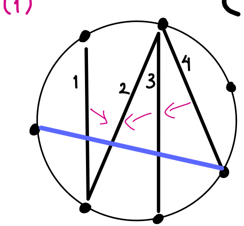
Fix an $(n+3)$-gon with a triangulation T.

diagonals not in T $\overset{1:1}{\longleftrightarrow}$ indecomposable representations

where T crosses

equivalent to polygon model for Conway-Coxeter friezes

$(4+3)$-gon



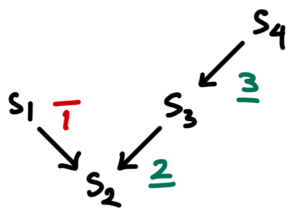
$1_2{}^3$

η surjection (Björner – Wachs 1997, Reading 2006)

Q quiver of type A_{n+2}

$s_1 — s_2 — \cdots — s_{n+2}$

$Q =$

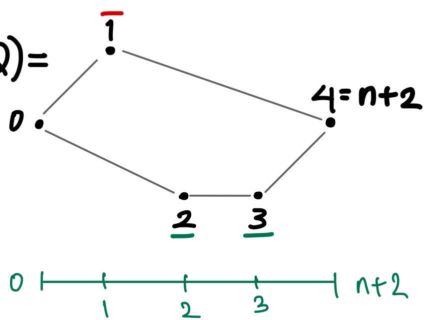


Type A_{2+2}

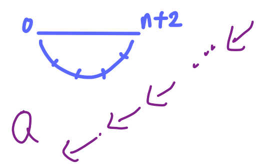
if $s_i \searrow s_{i+1}$ in Q, let $\bar{i}$ be up

if $s_i \swarrow s_{i+1}$ in Q, let $\underline{i}$ be down

$P(Q) =$

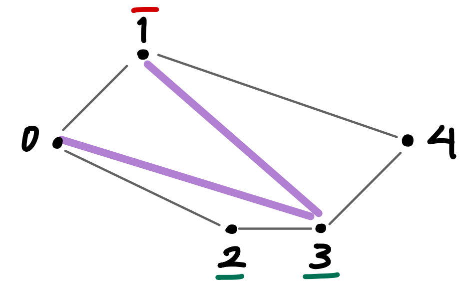


In Emily's lecture, all vertices are down



$\eta_Q : \underbrace{S_{n+1}}_{\text{symmetric group}} \twoheadrightarrow \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } P(Q) \end{array} \right\}$

$w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3 \quad \mapsto$



along top edges

paths from 0 to n+2

$\lambda_3 \quad 0 \ \bar{1} \ 4$ (remove 3) $\quad w(3) = \underline{3}$

$\lambda_2 \quad 0 \ \bar{1} \ \underline{3} \ 4$ (insert $\bar{1}$) $\quad w(2) = \bar{1}$

$\lambda_1 \quad 0 \ \underline{3} \ 4$ (remove 2) $\quad w(1) = \underline{2}$

$\lambda_0 \quad 0 \ \underline{2} \ \underline{3} \ 4$

along bottom edges

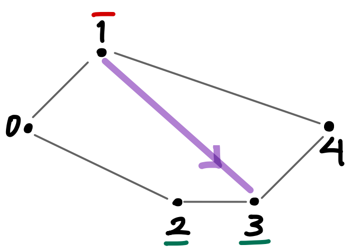
New model of rep Q (Barnard – G. – Meehan – Schiffler 2019)

Let $\mathcal{E} := \{$line segments $\gamma(i,j)$ for $0 \le i < j \le n+2\}$

Define a bijection $F : \mathcal{E} \longrightarrow \{$indecomposable representations of $A_{n+2}\}$

line segment $\gamma(i,j) \longrightarrow M(i+1, j)$

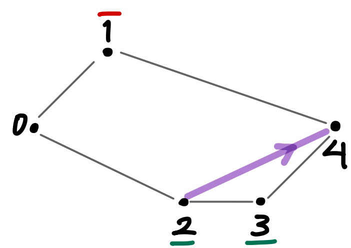
E.g.



$\gamma(1,3)$

$F \downarrow$

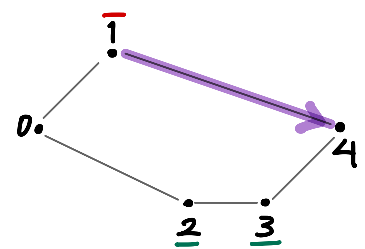
$M(2,3) = {}_{2}{}^{3}$



$\gamma(2,4)$

$F \downarrow$

$M(3,4) = {}_{3}{}^{4}$



$\gamma(1,4)$

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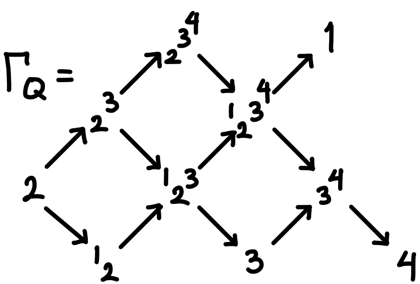
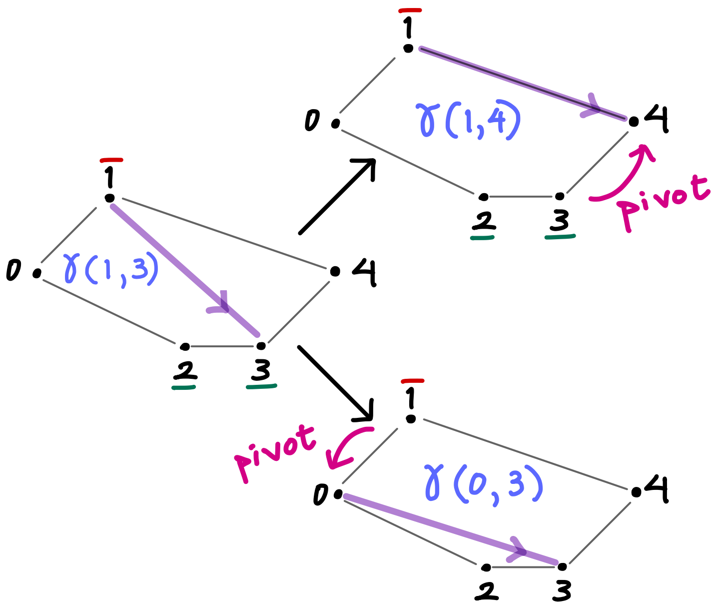
$M(2,4) = {}_{2}{}^{3^{4}}$

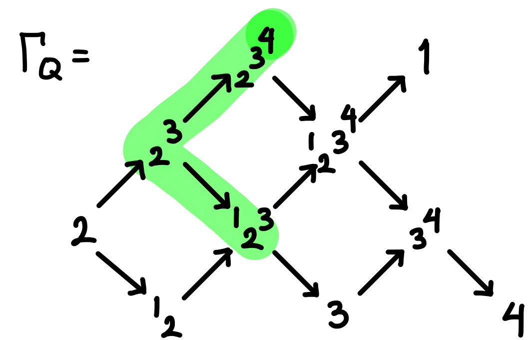
$\Gamma_Q =$



New model of rep Q (Barnard – G. – Meehan – Schiffler 2019)

<u>Thm</u> Every arrow (irreducible morphism) in $\underbrace{\Gamma_Q}_{\text{AR quiver}}$ acts on the line segments $\gamma(i,j)$ by pivoting one endpoint to its counterclock neighbor.





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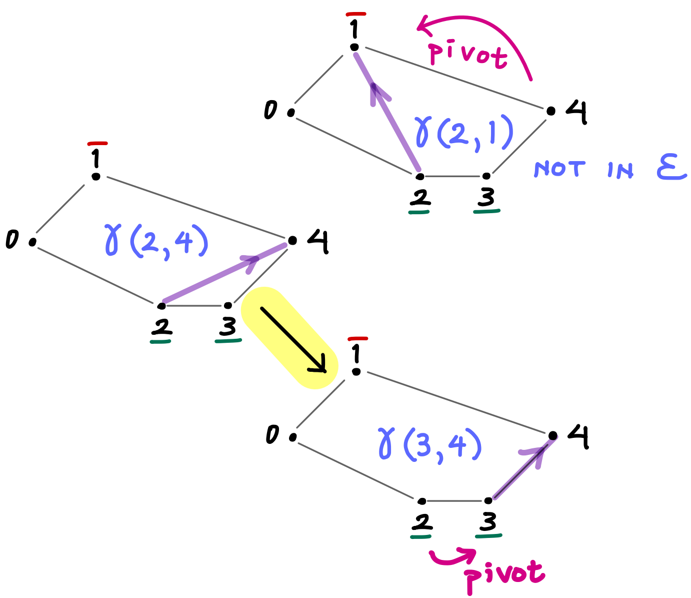
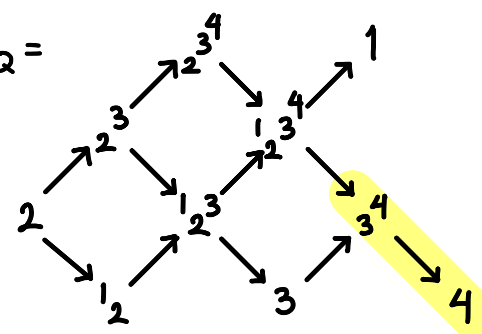
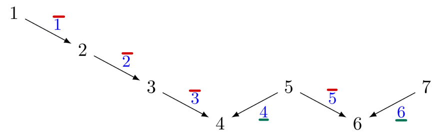


$\Gamma_Q =$

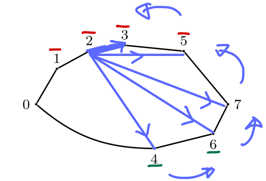


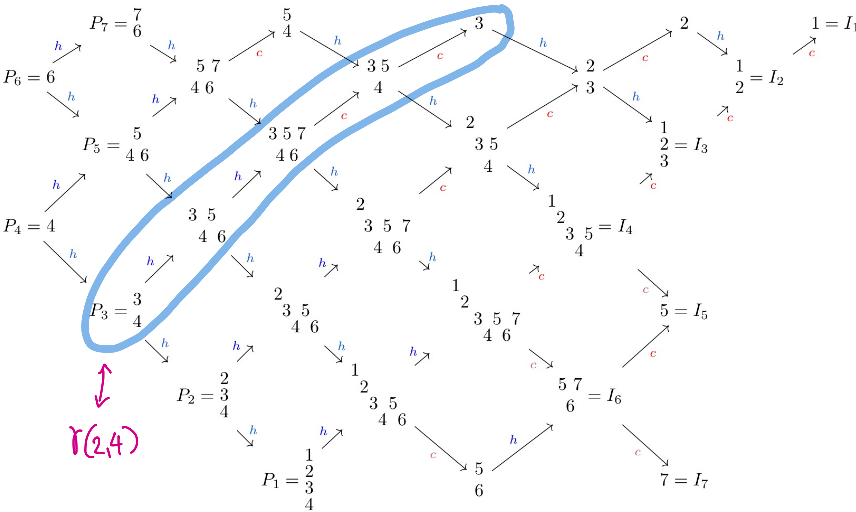
A_{5+2} example

$Q =$

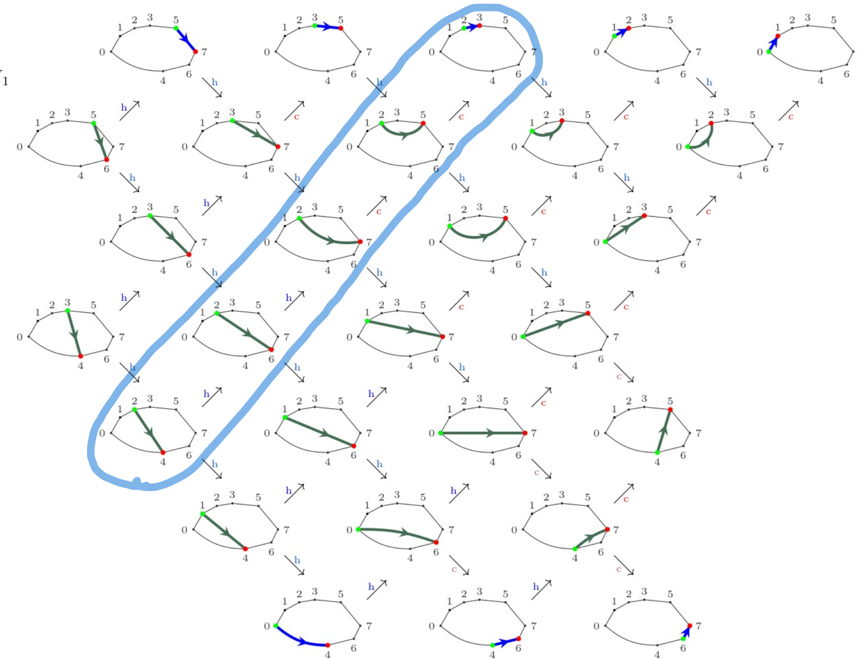


$P(Q)$



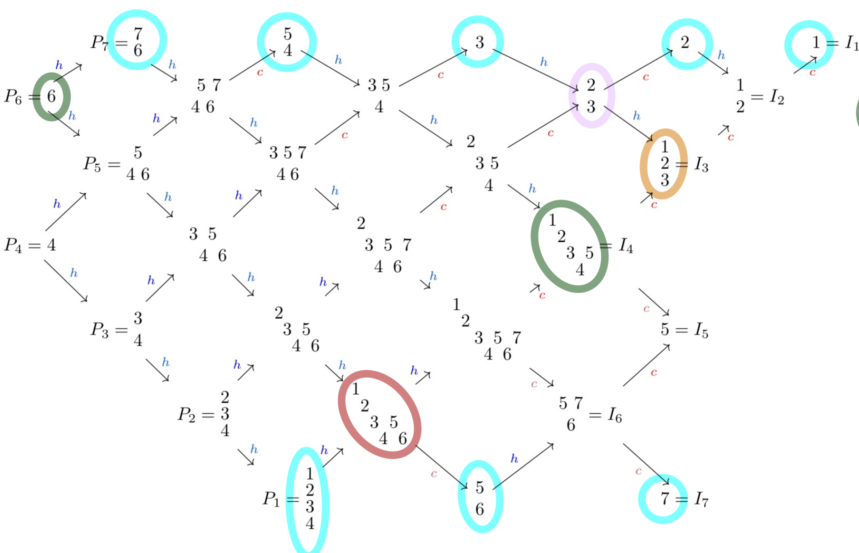


$\gamma(2,4)$



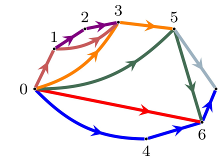
What does the triangulation in our model correspond to?

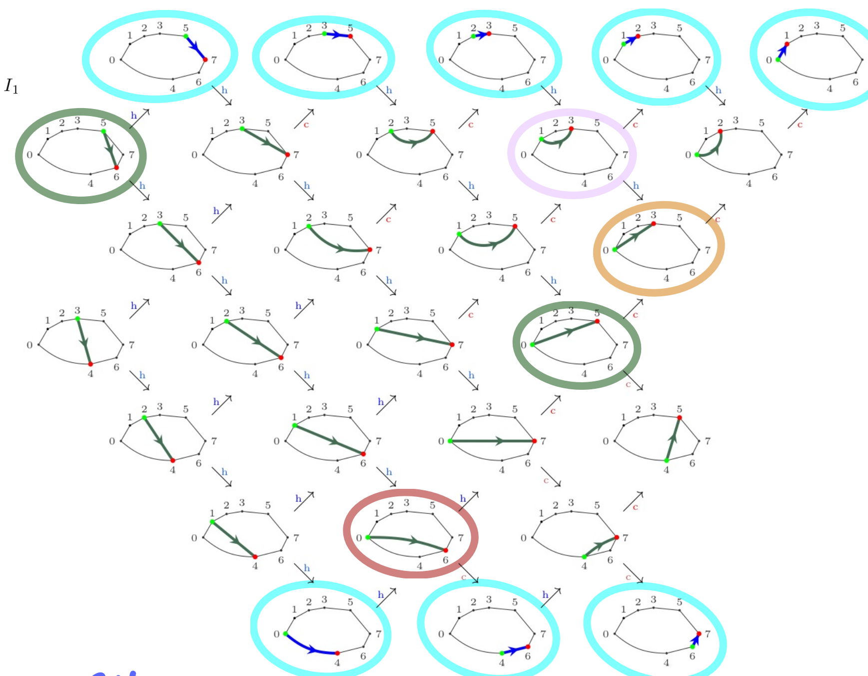
$$Q = \begin{tikzcd} 1 \arrow[r, "\overline{1}"] & 2 \arrow[r, "\overline{2}"] & 3 \arrow[r, "\overline{3}"] & 4 & 5 \arrow[l, "\underline{4}"'] \arrow[r, "\overline{5}"] & 6 & 7 \arrow[l, "\underline{6}"'] \end{tikzcd}$$



8+5=13 indecomposable summands

A triangulation in P(Q)





8+5=13 line segments

$\boxed{\mathrm{mar}(Q)}$ (Barnard–G.–Meehan–Schiffler)

<u>Thm</u> F gives a bijection $\left\{\begin{array}{c}\text{triangulations}\\ \text{of } P(Q)\end{array}\right\} \longrightarrow \underbrace{\left\{\begin{array}{c}\text{maximal almost}\\ \text{rigid representations}\end{array}\right\}}_{\mathrm{mar}(Q)}$

<u>Def</u> * T is **almost rigid** if

- T is basic (no repeated indecomposable summands)
- For each pair A, B of indecomposable summand of T,

  if $0 \to A \xrightarrow{f} E \xrightarrow{g} B \to 0$ is a short exact sequence

  $\mathrm{Im}\, f = \ker g$

  then $E \cong A \oplus B$ or E is indecomposable

We cannot have

$0 \to A \to \underbrace{C \oplus D}_{\ncong A \oplus B} \to B \to 0$

as a s.e.s

* An almost rigid T is **maximal almost rigid** if

$T \oplus M$ is not almost rigid for any representation M.

<u>Cor</u> Let Q be a type A_{n+2} quiver.

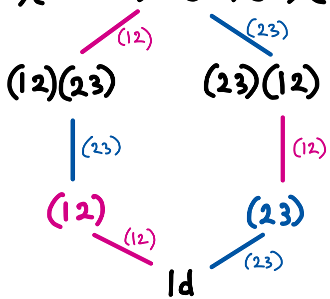
$\#\{\text{summands in } T \in \mathrm{mar}(Q)\} = 2n+3$ ($n+3$ boundary line segments, n internal diagonals)

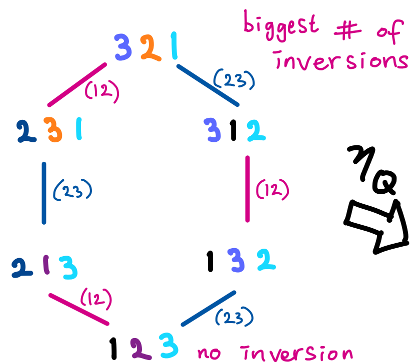
$\#\,\mathrm{mar}(Q) = \frac{1}{n+2}\binom{2n+2}{n+1}$ Catalan numbers!

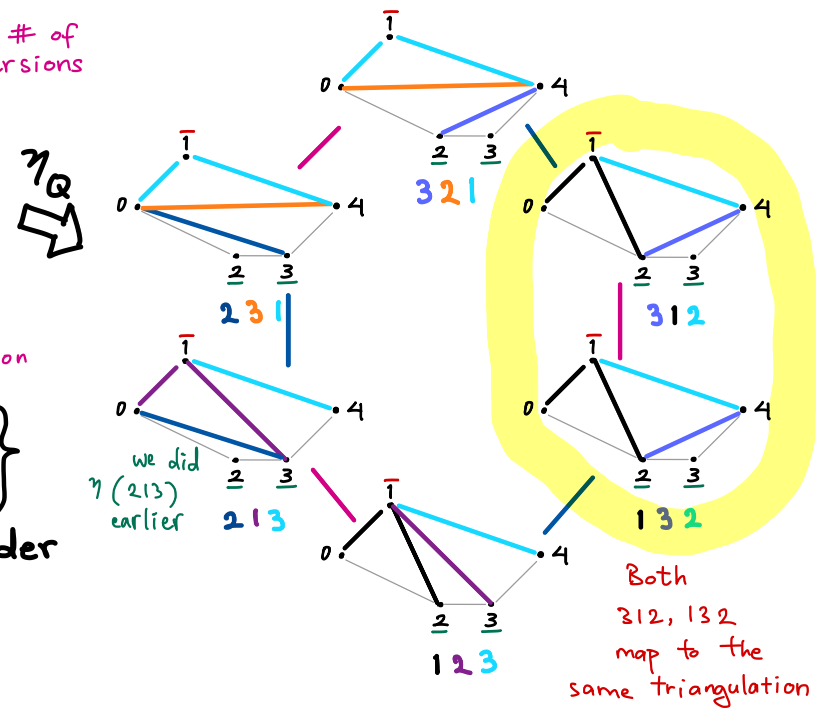
Cambrian lattice using η (Reading 2006)

The (right) weak order on S_{n+1} is a $\overbrace{\text{partial order}}^{\text{in fact, a lattice}}$ (poset) whose Hasse diagram is the Cayley graph of S_{n+1} with generators $\{(1,2),(2,3),\dots,(n,n+1)\}$

E.g. $(12)(23)(12)=(23)(12)(23)$







$$\eta_Q : S_{n+1} \xrightarrow[\text{from } 0 \text{ to } n+2]{\text{union of paths}} \left\{ \begin{array}{c} \text{triangulations} \\ \text{of} \quad P(Q) \end{array} \right\}$$

gives a quotient of the weak order called <u>Q-Cambrian lattice</u>.

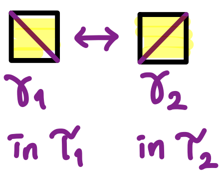
Cambrian lattice on triangulations of P(Q) (Reading 2006)

T_1 is covered by T_2

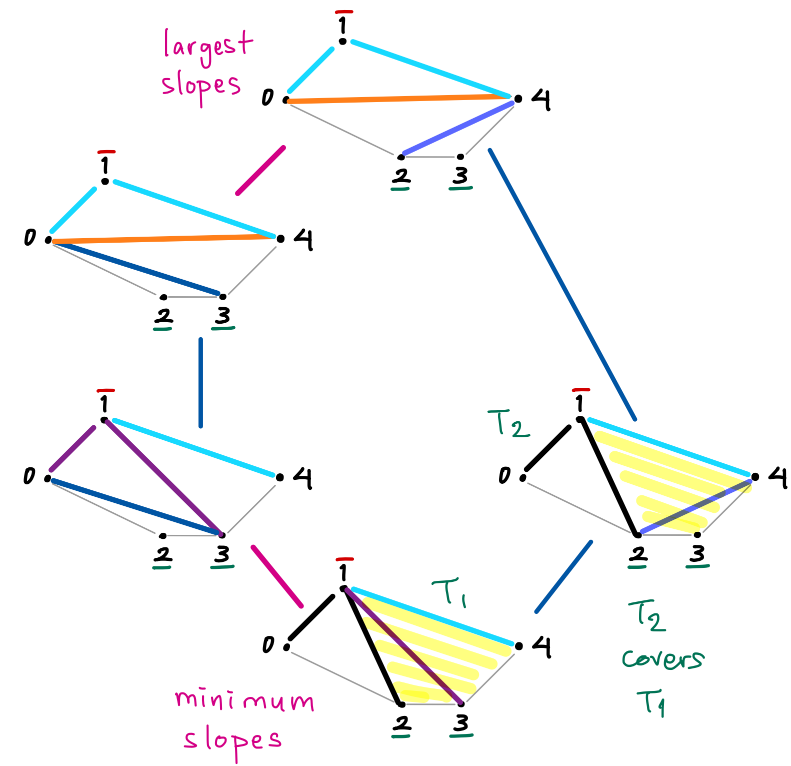
$T_1 \lessdot T_2$

$$\begin{array}{c} T_2 \\ | \\ T_1 \end{array} \quad \text{covering relation} \qquad \text{if:}$$

- T_1, T_2 differ by a diagonal flip



- The diagonal γ_2 has larger slope



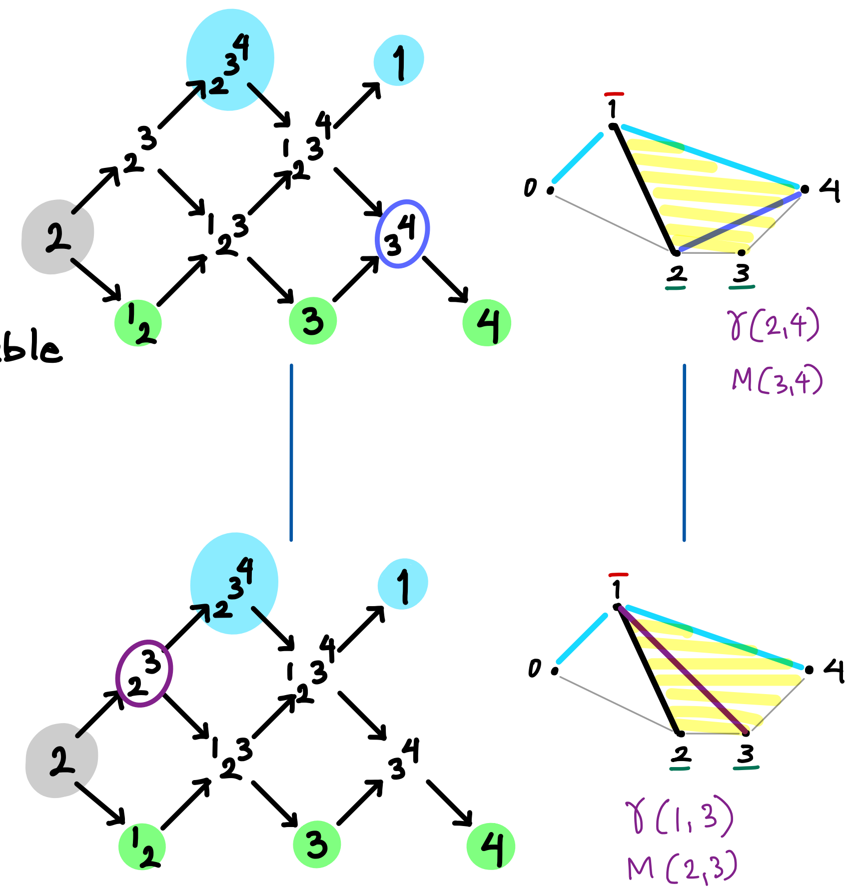
Cambrian lattice on $\mathrm{mar}(Q)$

(Barnard–G.–Meehan–Schiffler)

$$\begin{array}{c} T_2 \\ | \\ T_1 \end{array}$$ covering relation if :

- T_1, T_2 differ by one indecomposable summand $\underset{\text{in } T_1}{M_1} \sim \underset{\text{in } T_2}{M_2}$

- There is a nonzero morphism from M_1 to M_2

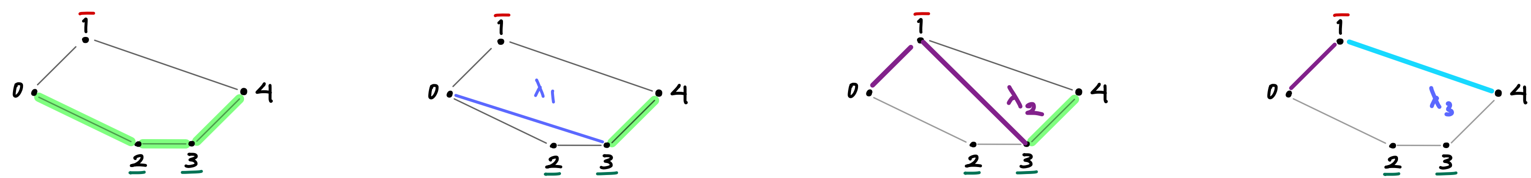


$$\eta : S_{n+1} \xrightarrow{\text{union of paths}} \left\{ \begin{array}{c}\text{triangulations} \\ \text{of } P(Q)\end{array} \right\} \quad \text{induces} \quad \eta^r : S_{n+1} \xrightarrow[\text{with dimension vector } (1,1,\dots,1)]{\text{"union" of representations}} \text{mar}(Q)$$

E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\qquad w(1) = \underline{2}$ $\qquad w(2) = \overline{1}$ $\qquad w(3) = \underline{3}$

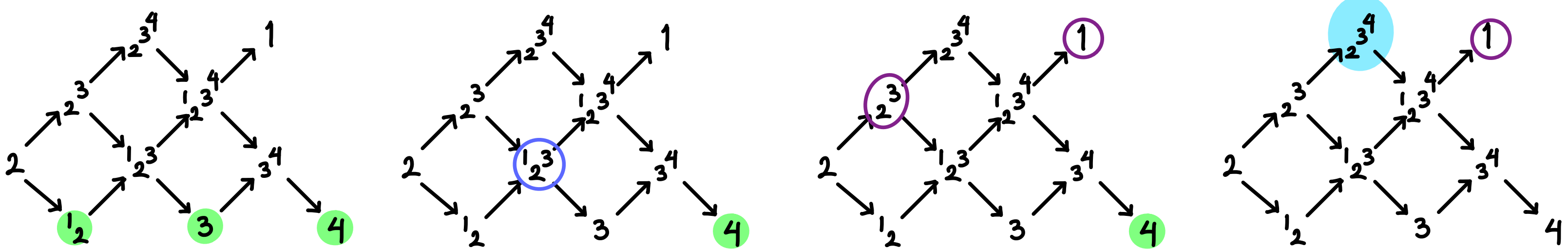
remove $\underline{2}$ $\qquad$ insert $\overline{1}$ $\qquad$ remove $\underline{3}$

$\lambda_0 \quad 0 \ \underline{2} \ \underline{3} \ 4$ $\qquad \lambda_1 \quad 0 \ \times \ \underline{3} \ 4$ $\qquad \lambda_2 \quad 0 \ \overline{1} \ \underline{3} \ 4$ $\qquad \lambda_3 \quad 0 \ \overline{1} \ \times \ 4$



Take union of all edges of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to get triangulation $\eta(213)$

$\lambda_0^r = {}^{1}_{2} \oplus 3 \oplus 4$ $\qquad \lambda_1^r = {}^{1}{}_{2}{}^{3} \oplus 4$ $\qquad \lambda_2^r = 1 \oplus {}_{2}{}^{3} \oplus 4$ $\qquad \lambda_3^r = 1 \oplus {}_{2}{}^{3}{}^{4}$



ext$(\underline{2})$ $\qquad$ deg$(\overline{1})$ $\qquad$ ext$(\underline{3})$

Take union of all circled indecomposables to get m.a.r rep $\eta^r(213)$

THANK

YOU!