Queen's University

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Talk notes at egunawan.github.io/talks/21/november/queensu21.pdf

Cambrian combinatorics on quiver representations (type $\mathbb{A}$)

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(Jt. with E. Barnard, E. Meehan, R. Schiffler)

(arXiv: 1912.02840)

Outline

1. $\eta : S_{n+1} \twoheadrightarrow \{\Delta\text{tions}\}$ $\underbrace{\text{Cambrian lattice}}$

   an orientation of the 1-skeleton of the associahedron / type $\mathbb{A}$ cluster algebra exchange graph oriented by green mutations / etc

2. η polygon model for rep Q

3. What do Δtions correspond to? $\underbrace{\text{Maximal almost rigid}}_{\text{mar}}$ representations

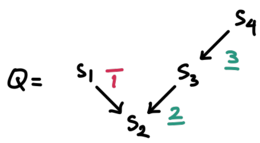
4. The endomorphism algebra of a mar representation is a type $\mathbb{A}$ tilted algebra.

5. $\eta^{\text{rep}}: S_{n+1} \twoheadrightarrow \text{mar}(Q)$ Cambrian lattice

1. Cambrian lattice from η (Reading 2004)

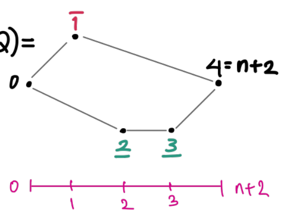
Q quiver of type A_{n+2}

$s_1 - s_2 - \cdots - s_{n+2}$

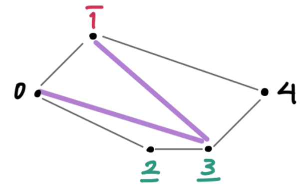


if $s_i \searrow s_{i+1}$ in Q, let $\bar{i}$ be up

if $s_i \swarrow s_{i+1}$ in Q, let $\underline{i}$ be down



$\eta_Q : \underbrace{S_{n+1}}_{\text{symmetric group}} \twoheadrightarrow \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } P(Q) \end{array} \right\}$

$w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3 \mapsto$ 

along top edges

paths from 0 to n+2

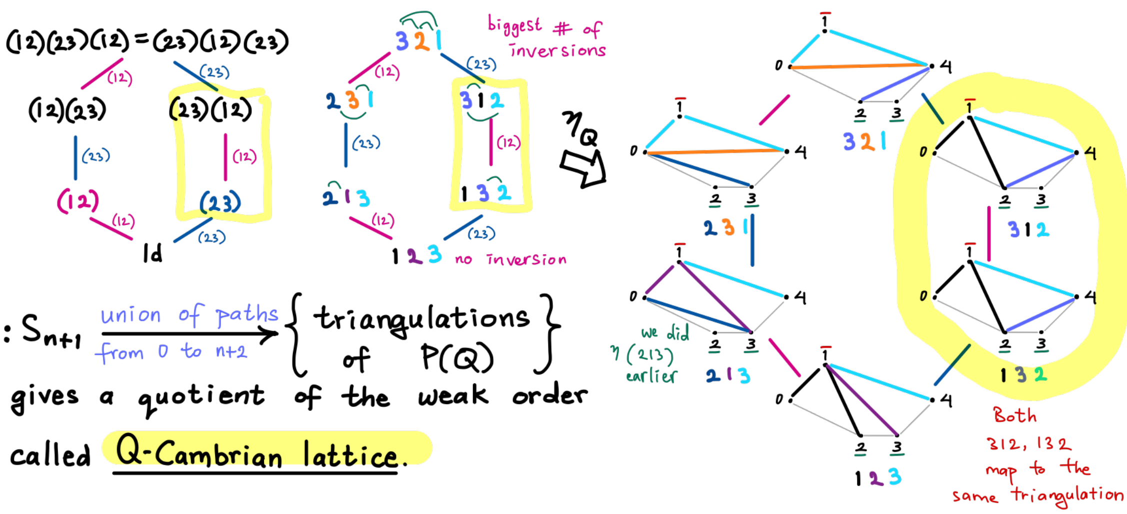
λ_3	$0\ \bar{1}\ \overset{\times}{\underset{\text{remove } \underline{3}}{}}\ 4$	$w(3) = \underline{3}$
λ_2	$0\ \bar{1}\ \underline{3}\ 4$ (insert $\bar{1}$)	$w(2) = \bar{1}$
λ_1	$0\ \overset{\times}{\underset{\text{remove } \underline{2}}{}}\ \underline{3}\ 4$	$w(1) = \underline{2}$
λ_0	$0\ \underline{2}\ \underline{3}\ 4$	

along bottom edges

The (right) weak order on S_{n+1} is a $\overbrace{\text{partial order (poset)}}^{\text{In fact, a lattice}}$ whose Hasse diagram is the Cayley graph of S_{n+1} with generators $\{(1,2), (2,3), \ldots, (n,n+1)\}$

E.g. $(12)(23)(12) = (23)(12)(23)$

S_3



$\eta_Q : S_{n+1} \xrightarrow[\text{from 0 to } n+2]{\text{union of paths}} \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } P(Q) \end{array} \right\}$

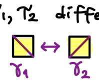
gives a quotient of the weak order called **Q-Cambrian lattice**.

$\begin{array}{c} T_2 \\ | \\ T_1 \end{array}$ covering relation if:

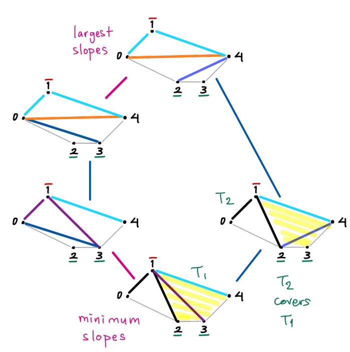
T_1 is covered by T_2

$T_1 \lessdot T_2$

- T_1, T_2 differ by a diagonal flip



- The diagonal γ_2 has larger slope



2. η polygon model for rep Q

Special case of: For gentle A,

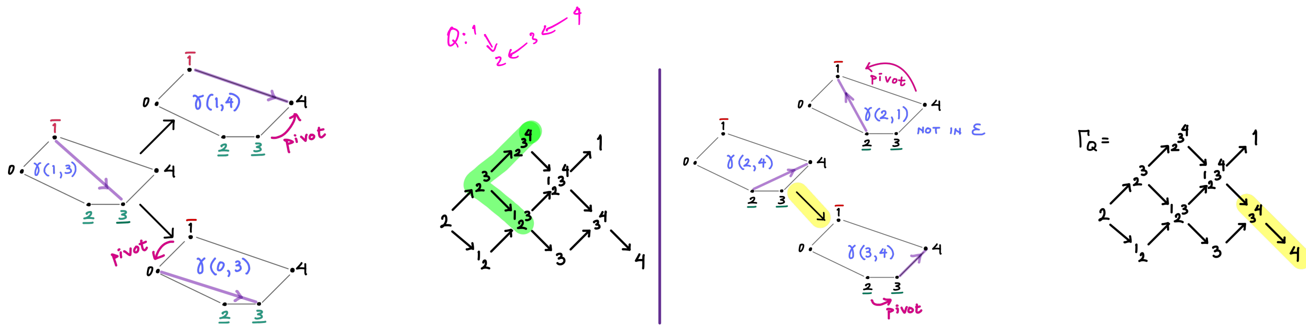
- Baur – Coelho Simões 2018 model for mod A
- Opper-Plamondon-Schroll 2018 model for D^b (mod A)

$\mathcal{E} := \{$ oriented line segments $\gamma(i,j) \mid 0 \leq i < j \leq n+2\}$

$\mathcal{C}_{P(Q)}$ category whose objects are $\mathcal{E}$

gentle

irreducible morphisms are pivoting one endpoint counterclockwise



Thm A There is an equivalence of categories

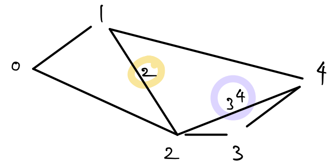
$\mathcal{C}_{P(Q)} \longrightarrow \text{Ind } Q$

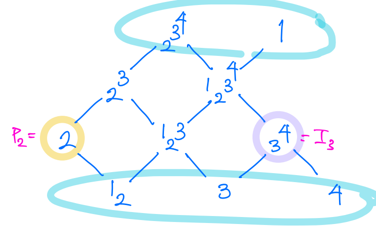
$\gamma(i-1, j) \longleftrightarrow M(i, j) = \cdots 0 - \overset{i}{k} \overset{1}{-} \cdots \overset{1}{-} \overset{j}{k}$

endpoint pivots $\longleftrightarrow$ irreducible morphisms

clockwise rotation $\longleftrightarrow$ AR translation τ

3. Δtion corresponds to ... ?





<u>Recall Def</u> T is <u>tilting</u> if

- T is basic (no repeated indecomposable summands)
- For each pair A, B of indecomposable summands of T, every short exact sequence is split.

<u>Def</u> T is <u>almost rigid</u> if

- T is basic (no repeated indecomposable summands)
- For each pair A, B of indecomposable summands of T, if $0 \to A \to E \to B \to 0$ is a non-split s.e.s then E is indecomposable.
- An almost rigid T is <u>maximal almost rigid</u> if $T \oplus M$ is not almost rigid for any representation M.

<u>Thm B</u> $\left\{ \begin{array}{c} \Delta\text{tions} \\ \text{of } P(Q) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{maximal almost} \\ \text{rigid representations of } Q \end{array} \right\}$

$\mathrm{mar}(Q) :=$

<u>Cor</u> Q type A_{n+2} quiver, $T \in \mathrm{mar}(Q)$

$\# \{\text{summands of } T\} = 2n+3$ ($n+3$ boundary line segments, n internal — " —)

$\# \,\mathrm{mar}(Q) = \dfrac{1}{n+2}\dbinom{2n+2}{n+1}$ Catalan numbers! :)

<u>Work in-progress</u> For representation-finite gentle algebra kQ/I,

$\left\{ \begin{array}{c} \text{permissible } \Delta\text{tions} \\ \text{of } (\underbrace{S, M}_{\text{marked surface}}, \underbrace{\Gamma}_{\text{ribbon graph}}) \end{array} \right\} \longleftrightarrow \mathrm{mar}(Q, I)$

using [Baur – Coelho Simões] + [Opper–Plamondon–Schroll] model

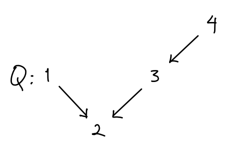
$\boxed{4. \quad \operatorname{End}_{\operatorname{rep} Q} T \quad \text{for} \quad T \in \operatorname{mar}(Q)}$

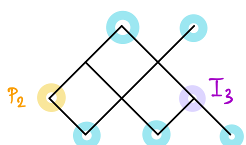
Def (Happel–Ringel 1982) Let M be a tilting module in $\operatorname{rep} \overline{Q}$.

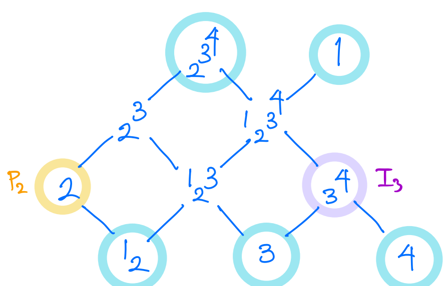
$\operatorname{End}_{\operatorname{rep} \overline{Q}} M$ is called a <u>tilted algebra of type $\overline{Q}$</u>.

Thm C Let $T \in \overset{A_{n+2}}{\operatorname{mar}(Q)}$, $C := \operatorname{End}_{\operatorname{rep} Q}(T)$.

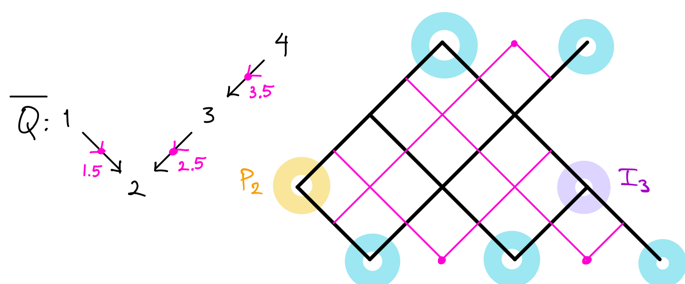
Then C is a tilted algebra of type $\underset{\substack{A_{2n+3} \\ n+2+n+1}}{\overline{Q}}$

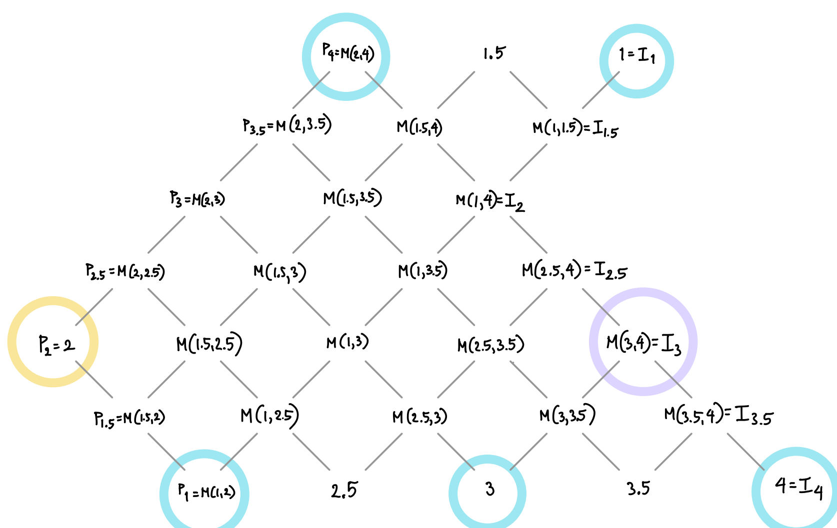






Summands of T are circled





Summands of $\overline{T}$ are circled

$$\underset{\substack{\downarrow \\ \text{mar}}}{\operatorname{End}_{\operatorname{rep} Q} T} \cong \operatorname{End}_{\operatorname{rep} \overline{Q}} \underset{\substack{\downarrow \\ \text{tilting in rep } \overline{Q}}}{\overline{T}}$$

5. $\eta^{rep}: S_{n+1} \twoheadrightarrow \mathrm{mar}(Q)$ Cambrian lattice

Def poset on $\mathrm{mar}(Q)$

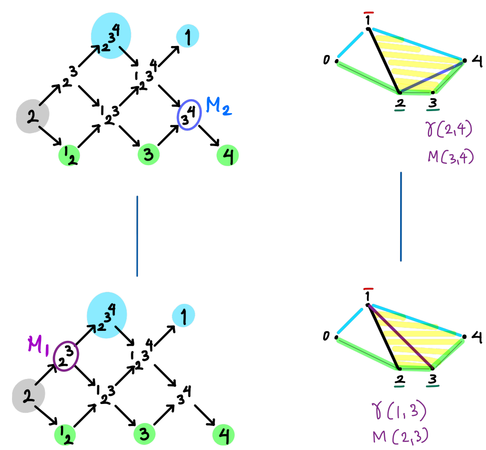
T_2 covering relation T_1 if:

(1) T_1, T_2 differ by one indecomposable summand $\underset{\text{in } T_1}{M_1} \sim \underset{\text{in } T_2}{M_2}$

(2) There is a short exact sequence

$$0 \to M_1 \to A \oplus B \to M_2 \to 0$$

where A, B are indecomposable summands of T_1 / M_1



Thm D The above poset on $\mathrm{mar}(Q)$ is a Cambrian lattice.

Rem The minimum mar module contains all projectives.

The maximum mar module contains all injectives.

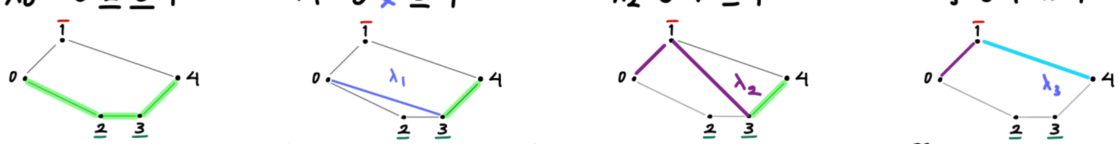
$\eta: S_{n+1} \xrightarrow{\text{union of paths}} \left\{ \begin{matrix} \text{triangulations} \\ \text{of } P(Q) \end{matrix} \right\}$ induces $\eta^r: S_{n+1} \xrightarrow[\text{with dimension vector } (1,1,\dots,1)]{\text{"union" of representations}} \mathrm{mar}(Q)$

dim

E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\quad w(1) = \underline{2}$ $\quad w(2) = \overline{1}$ $\quad w(3) = \underline{3}$

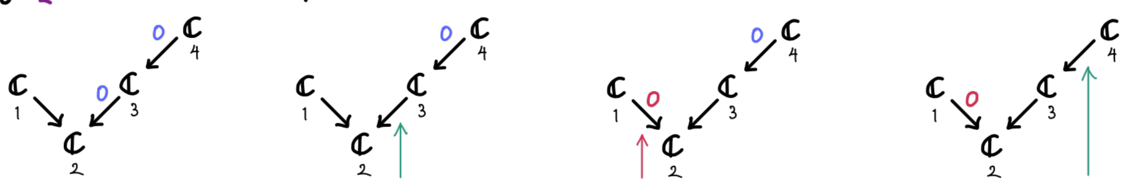
remove $\underline{2}$ $\quad$ insert $\overline{1}$ $\quad$ remove $\underline{3}$

$\lambda_0 \ \ 0\ \underline{2}\ \underline{3}\ 4$ $\quad \lambda_1 \ \ 0\ \times\ \underline{3}\ 4$ $\quad \lambda_2 \ \ 0\ \overline{1}\ \underline{3}\ 4$ $\quad \lambda_3 \ \ 0\ \overline{1}\ \times\ 4$



Take union of all edges of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to get triangulation $\eta(213)$

$\lambda_0^r = {}^1_2 \oplus 3 \oplus 4$ $\quad \lambda_1^r = {}^1{}_2{}^3 \oplus 4$ $\quad \lambda_2^r = 1 \oplus {}_2{}^3 \oplus 4$ $\quad \lambda_3^r = 1 \oplus {}_2{}^{3^4}$



ext($\underline{2}$) $\quad$ deg($\overline{1}$) $\quad$ ext($\underline{3}$)

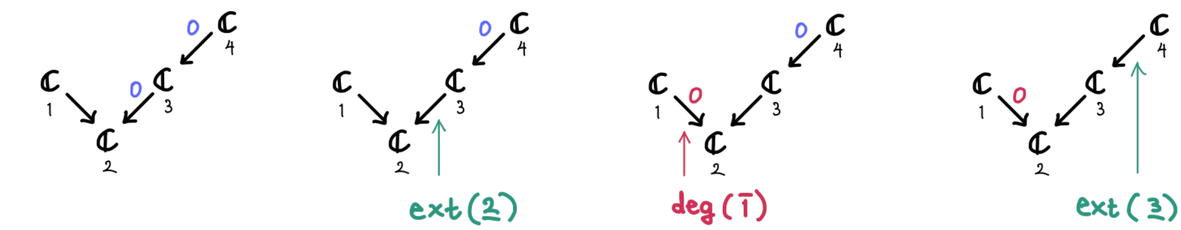
Take union of all 7 indecomposable summands to get m.a.r rep $\eta^r(213)$

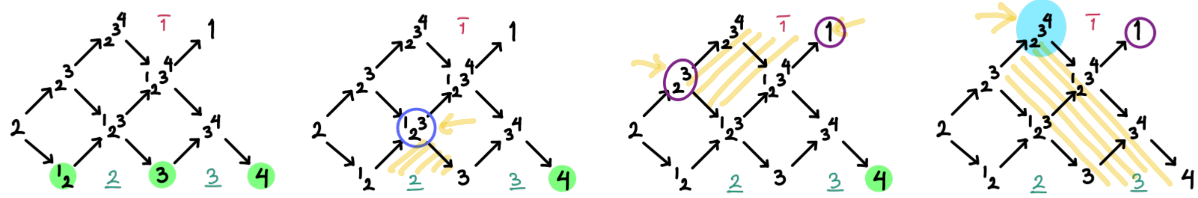
(Extra page)

$\eta^r : S_{n+1} \xrightarrow{\text{"union" of representations}} \mathrm{mar}(Q)$ via "rectangles with missing corners"

E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\qquad w(1) = \underline{2} \qquad w(2) = \bar{1} \qquad w(3) = \underline{3}$

$\lambda_0^r = {}^1_2 \oplus 3 \oplus 4 \qquad \lambda_1^r = {}^1{}_2{}^3 \oplus 4 \qquad \lambda_2^r = 1 \oplus {}_2{}^3 \oplus 4 \qquad \lambda_3^r = 1 \oplus {}_2{}^{3^4}$





Take union of all 7 circled indecomposables to get m.a.r rep $\eta^r(213)$