Cambrian combinatorics on quiver representations (type A)

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Emily Gunawan (University of Oklahoma)

Jt. with E. Barnard, E. Meehan, R. Schiffler

Type A quiver representations

Q an orientation of A_n Dynkin diagram

$$s_1 - s_2 - \cdots - s_n$$

rep Q : objects = finite-dimensional representations of Q

$\mathbb{C}Q$: path algebra
rep $Q \cong$ mod $\mathbb{C}Q$

morphisms = representation maps

Indecomposable representations

not a direct sum of two nonzero representations

are of the form

$$\begin{array}{c} 0 - 0 \cdots - \mathbb{C} \overset{\mathrm{Id}}{-} \mathbb{C} \cdots \overset{\mathrm{Id}}{-} \mathbb{C} - 0 \cdots 0 - 0 \\ \qquad\qquad\quad s_i \qquad\qquad\quad\; s_j \qquad\qquad\qquad \end{array}$$

$$M(i,j), \quad i \leq j$$

$\overset{1:1}{\longleftrightarrow}$ positive roots of type A_n

Example

A_4

$$Q = \begin{tikzcd}[row sep=small, column sep=small] & & & s_4 \arrow[dl] \\ s_1 \arrow[dr] & & s_3 \arrow[dl] & \\ & s_2 & & \end{tikzcd}$$

$$M(1,3) = \begin{tikzcd}[row sep=small, column sep=small] & & & 0 \arrow[dl, "0"'] \\ \mathbb{C} \arrow[dr, "\mathrm{Id}"] & & \mathbb{C} \arrow[dl, "\mathrm{Id}"'] & \\ & \mathbb{C} & & \end{tikzcd}$$

Shorthand notation $\begin{array}{ccc} 1 & & 3 \\ & 2 & \end{array}$

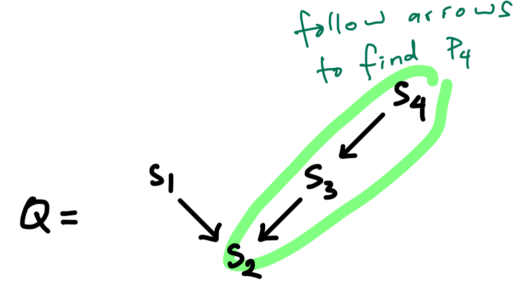
$$\alpha_1 + \alpha_2 + \alpha_3$$

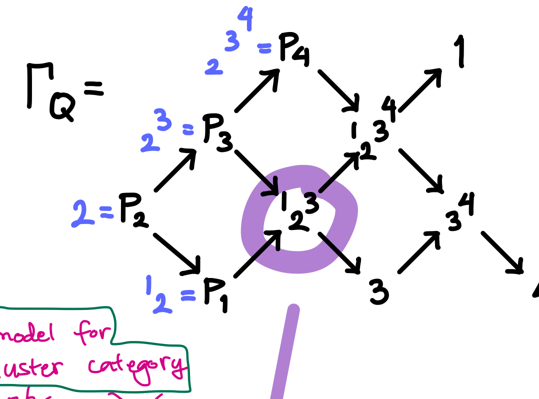
Auslander – Reiten quiver

The Auslander – Reiten quiver Γ_Q of rep Q is a connected directed graph with

vertices = indecomposable representations

arrows = irreducible morphisms — does not factor through another representation





model for cluster category $\mathcal{D}^b(\text{rep } Q) / \tau^{-1}[1]$

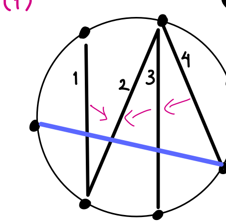
Caldero – Chapoton – Schiffler model 2006

Fix an $(n+3)$-gon with a triangulation T.

diagonals not in T $\overset{1:1}{\longleftrightarrow}$ indecomposable representations

where T crosses

(4+3)-gon

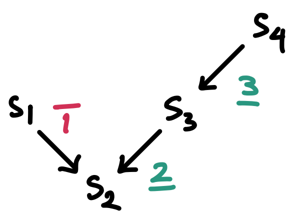


${}^1{}_2{}^3$

η surjection (Björner – Wachs 1997, Reading 2006)

Q quiver of type A_{n+2}

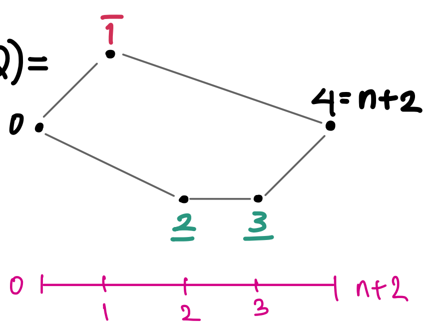
$s_1 - s_2 - \cdots - s_{n+2}$

$Q=$ 

Type A_{2+2}

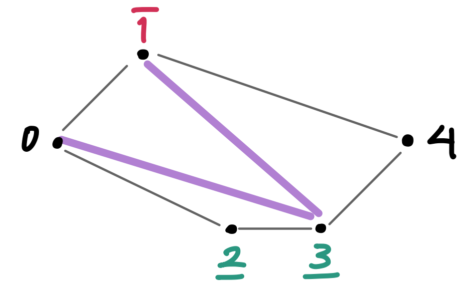
if $s_i \to s_{i+1}$ in Q, let $\bar{i}$ be up

if $s_i \leftarrow s_{i+1}$ in Q, let $\underline{i}$ be down

$P(Q)=$ 

$$\eta_Q : \underbrace{S_{n+1}}_{\text{symmetric group}} \twoheadrightarrow \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } P(Q) \end{array} \right\}$$

$$w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3 \mapsto$$



along top edges

paths from 0 to n+2

λ_3	$0\ \bar{1}\ \times\ 4$ (remove 3)	$w(3)=\underline{3}$
λ_2	$0\ \bar{1}\ \underline{3}\ 4$ (insert $\bar{1}$)	$w(2)=\bar{1}$
λ_1	$0\ \times\ \underline{3}\ 4$ (remove 2)	$w(1)=\underline{2}$
λ_0	$0\ \underline{2}\ \underline{3}\ 4$	

along bottom edges

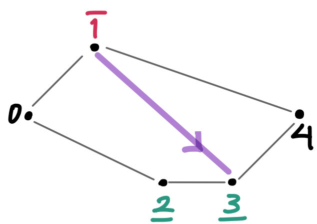
η polygon model of rep Q (Barnard–G.–Meehan–Schiffler 2019)

Let $\mathcal{E} := \{$oriented line segments $\gamma(i,j)$ for $0 \le i < j \le n+2\}$

Define a bijection $F : \mathcal{E} \longrightarrow \{$indecomposable representations of $A_{n+2}\}$

line segment $\gamma(i,j) \longrightarrow M(i+1, j)$

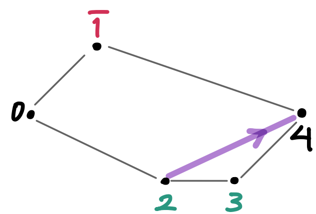
E.g.



$\gamma(1,3)$

$F \downarrow$

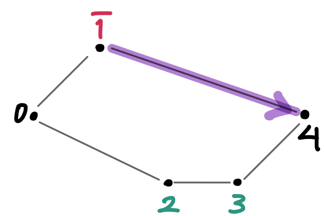
$M(2,3) = {}_{2}{}^{3}$



$\gamma(2,4)$

$F \downarrow$

$M(3,4) = {}_{3}{}^{4}$

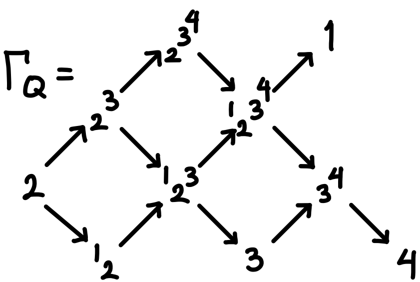


$\gamma(1,4)$

$F \downarrow$

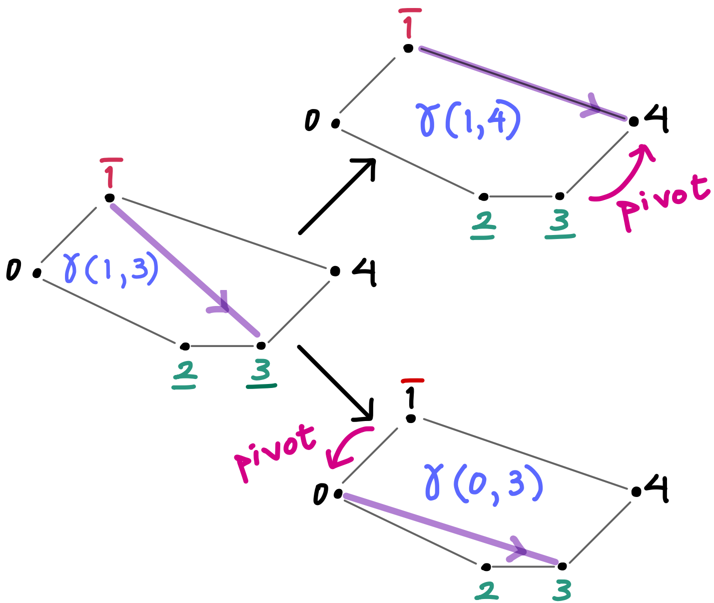
$M(2,4) = \begin{smallmatrix} & & 4 \\ & 3 & \\ 2 & & \end{smallmatrix}$

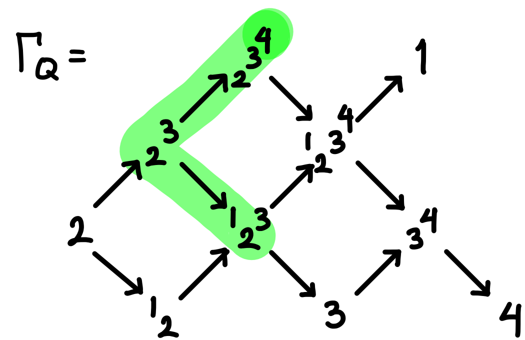
$\Gamma_Q =$



η polygon model of rep Q (Barnard–G.–Meehan–Schiffler 2019)

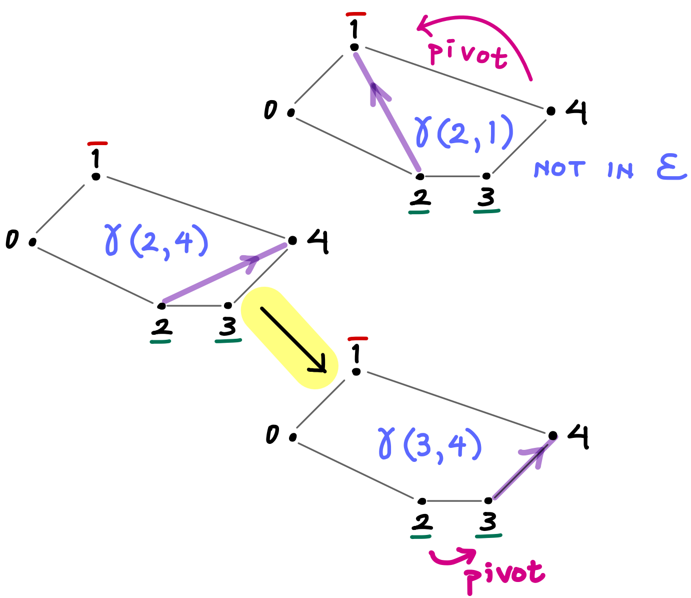
<u>Thm</u> Every arrow (irreducible morphism) in $\underbrace{\Gamma_Q}_{\text{AR quiver}}$ acts on the line segments $\gamma(i,j)$ by pivoting one endpoint to its counterclock neighbor.

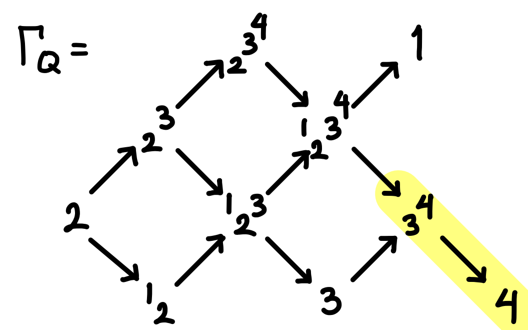




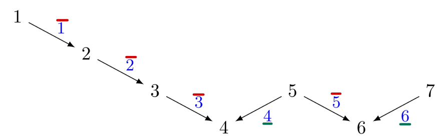
η polygon model of rep Q (Barnard–G.–Meehan–Schiffler 2019)

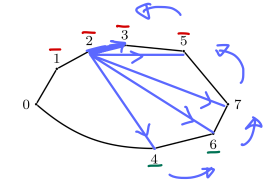
<u>Thm</u> Every arrow (irreducible morphism) in Γ_Q acts on the line segments $\gamma(i,j)$ by pivoting one endpoint to its counterclock neighbor.

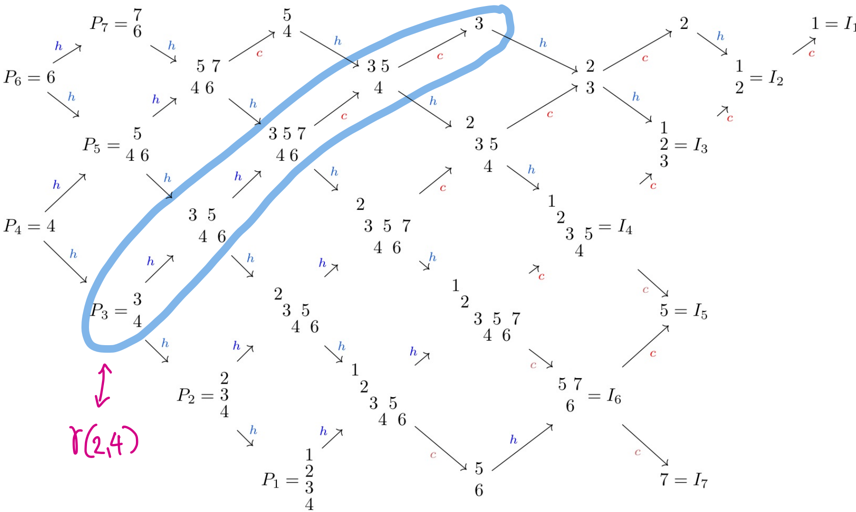




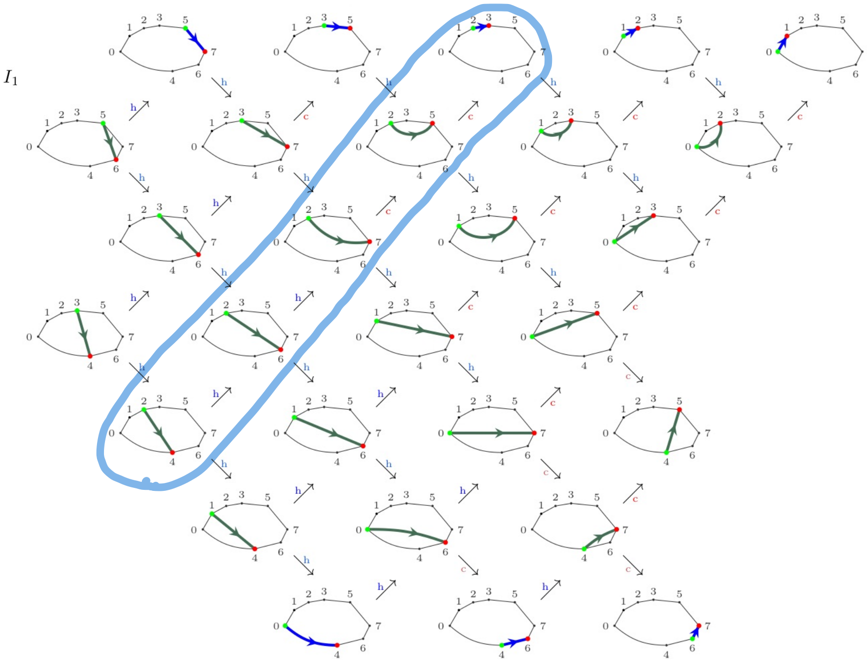
A_{5+2} example

$Q =$ 

$P(Q)$ 

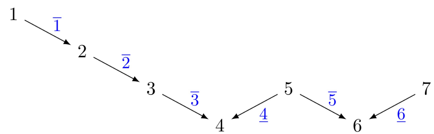


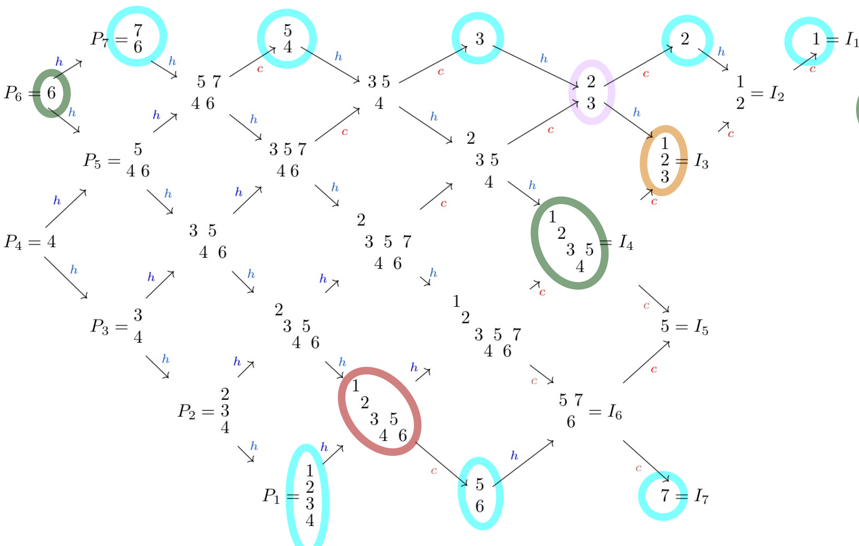
$\gamma(2,4)$



(Pause for questions)

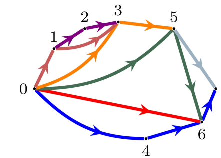
What does the triangulation in our model correspond to?

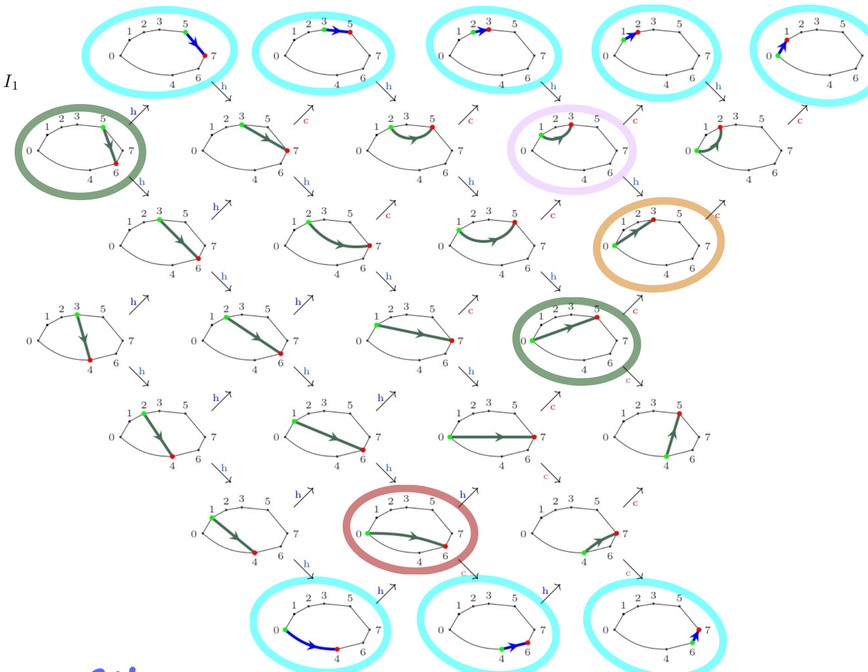
Q = 



8+5=13 indecomposable summands

A triangulation in P(Q)





8+5=13 line segments

$\boxed{\text{mar}(Q)}$ (Barnard–G.–Meehan–Schiffler)

<u>Thm</u> F gives a bijection $\left\{ \begin{matrix} \text{triangulations} \\ \text{of } P(Q) \end{matrix} \right\} \longrightarrow \underbrace{\left\{ \begin{matrix} \text{maximal almost} \\ \text{rigid representations} \end{matrix} \right\}}_{\text{mar}(Q)}$

<u>Def</u> * T is **almost rigid** if

- T is basic (no repeated indecomposable summands)
- For each pair A, B of indecomposable summands of T,

  if $0 \to A \to E \to B \to 0$ is a short exact sequence

  then $E \cong A \oplus B$ or E is indecomposable

We cannot have
$0 \to A \to \underbrace{C \oplus D}_{\ncong A \oplus B} \to B \to 0$
as a s.e.s

* An almost rigid T is **maximal almost rigid** if

  $T \oplus M$ is not almost rigid for any representation M.

<u>Cor</u> Let Q be a type A_{n+2} quiver and $T \in \text{mar}(Q)$

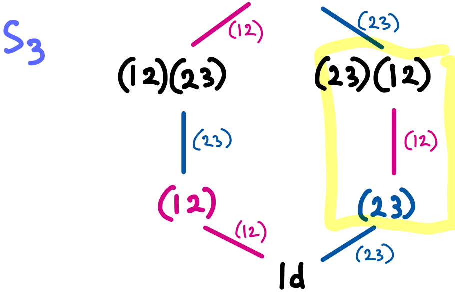
$\#\{\text{summands in } T\} = 2n+3$ (n+3 boundary line segments, n internal diagonals)

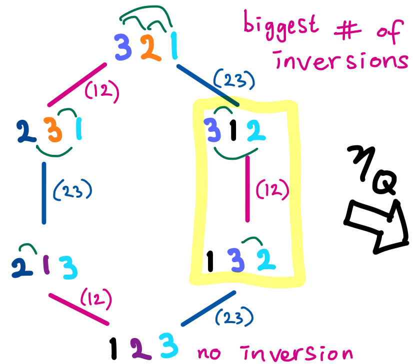
$\#\,\text{mar}(Q) = \frac{1}{n+2}\binom{2n+2}{n+1}$ Catalan numbers !

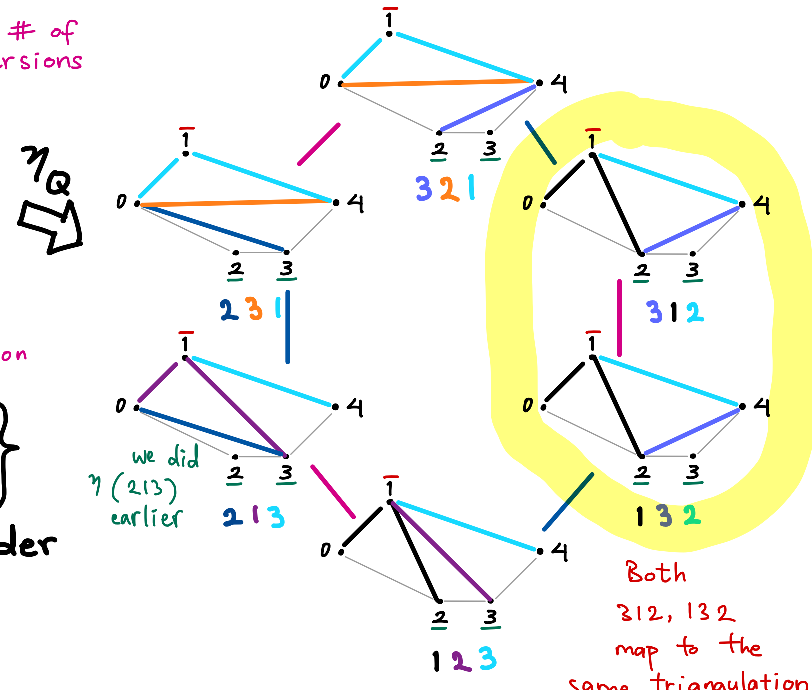
Cambrian lattice using η (Reading 2006)

The (right) weak order on S_{n+1} is a $\overbrace{\text{partial order (poset)}}^{\text{In fact, a lattice}}$ whose Hasse diagram is the Cayley graph of S_{n+1} with generators $\{(1,2),(2,3),\dots,(n,n+1)\}$

E.g. $(12)(23)(12) = (23)(12)(23)$







$$\eta_Q : S_{n+1} \xrightarrow[\text{from } 0 \text{ to } n+2]{\text{union of paths}} \left\{ \begin{array}{c} \text{triangulations} \\ \text{of } P(Q) \end{array} \right\}$$

gives a quotient of the weak order called <u>Q-Cambrian lattice</u>.

Cambrian lattice on triangulations of $P(Q)$ (Reading 2006)

T_1 is covered by T_2

$T_1 \lessdot T_2$

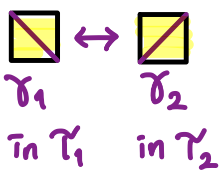
T_2

| covering relation

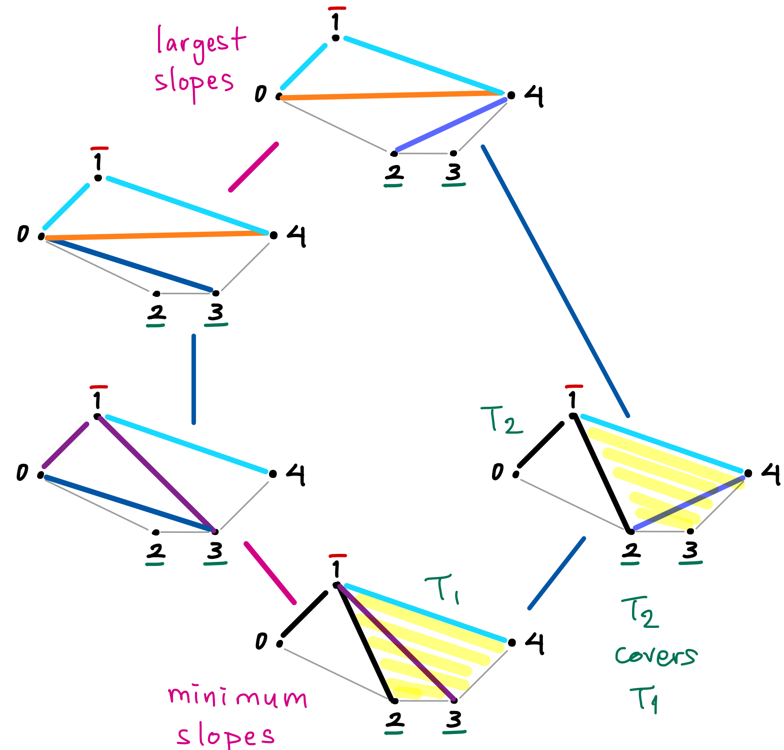
T_1

if:

- T_1, T_2 differ by a diagonal flip



- The diagonal γ_2 has larger slope



Cambrian lattice on mar(Q)

(Barnard – G. – Meehan – Schiffler)

T_2
|
T_1

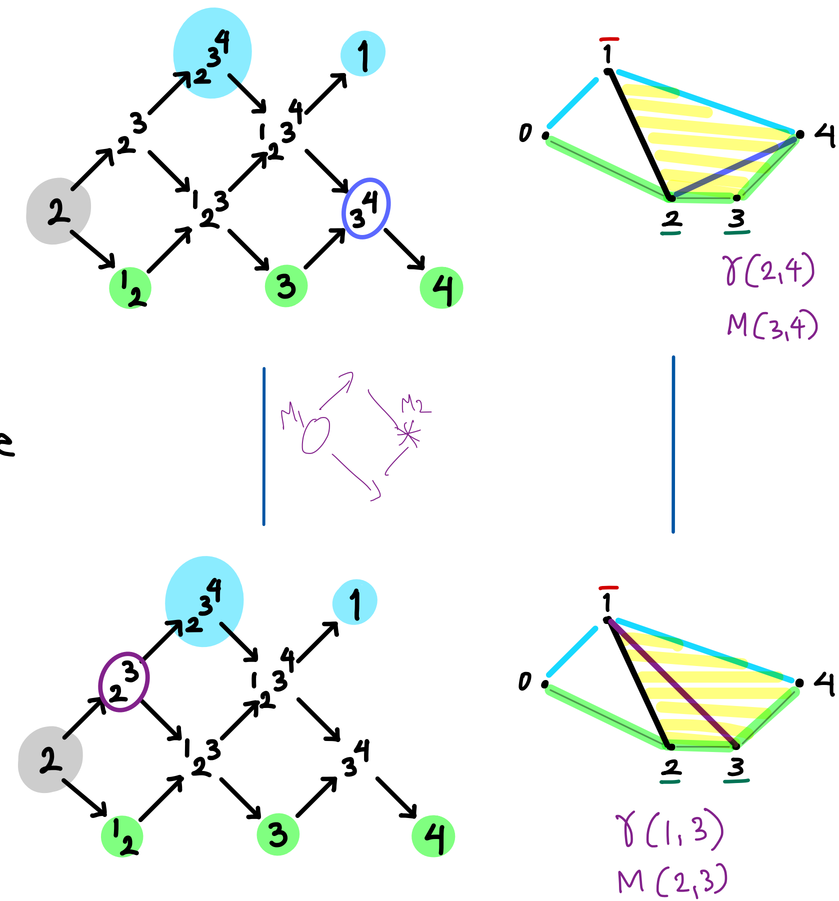
covering relation if :

(1) T_1, T_2 differ by one indecomposable summand $\underset{\text{in } T_1}{M_1} \sim \underset{\text{in } T_2}{M_2}$

(2) There is a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow A \oplus B \longrightarrow M_2 \longrightarrow 0$$

where A, B are indecomposable summands of T_1 / M_1



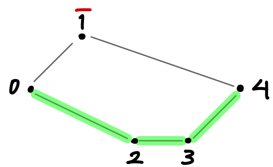
$\eta: S_{n+1} \xrightarrow{\text{union of paths}} \left\{\begin{matrix}\text{triangulations} \\ \text{of } P(Q)\end{matrix}\right\}$ induces $\eta^r: S_{n+1} \xrightarrow{\text{"union" of representations}} \mathrm{mar}(Q)$

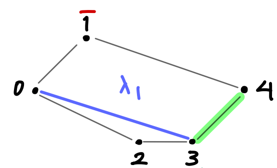
with $\underbrace{\text{dimension vector}}_{\dim}$ $(1,1,\dots,1)$

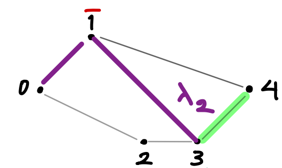
E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\quad w(1) = \underline{2}$ $\quad w(2) = \bar{1}$ $\quad w(3) = \underline{3}$

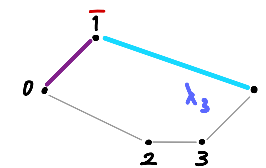
remove $\underline{2}$ $\quad$ insert $\bar{1}$ $\quad$ remove $\underline{3}$

$\lambda_0 \quad 0\ \underline{2}\ \underline{3}\ 4$ $\qquad \lambda_1 \quad 0\ \times\ \underline{3}\ 4$ $\qquad \lambda_2 \quad 0\ \bar{1}\ \underline{3}\ 4$ $\qquad \lambda_3 \quad 0\ \bar{1}\ \times\ 4$



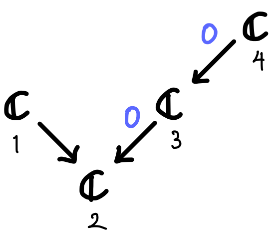


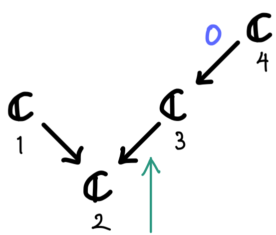


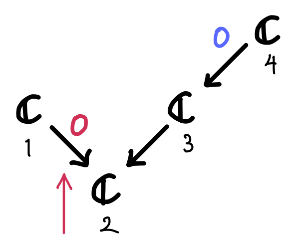


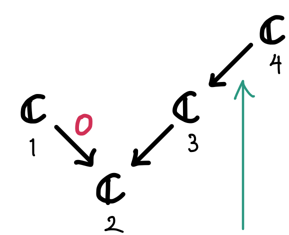
Take union of all edges of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ to get triangulation $\eta(213)$

$\lambda_0^r = {}^{1}_{2} \oplus 3 \oplus 4$ $\qquad \lambda_1^r = {}^{1}{}_{2}{}^{3} \oplus 4$ $\qquad \lambda_2^r = 1 \oplus {}_{2}{}^{3} \oplus 4$ $\qquad \lambda_3^r = 1 \oplus {}_{2}{}^{3}{}^{4}$









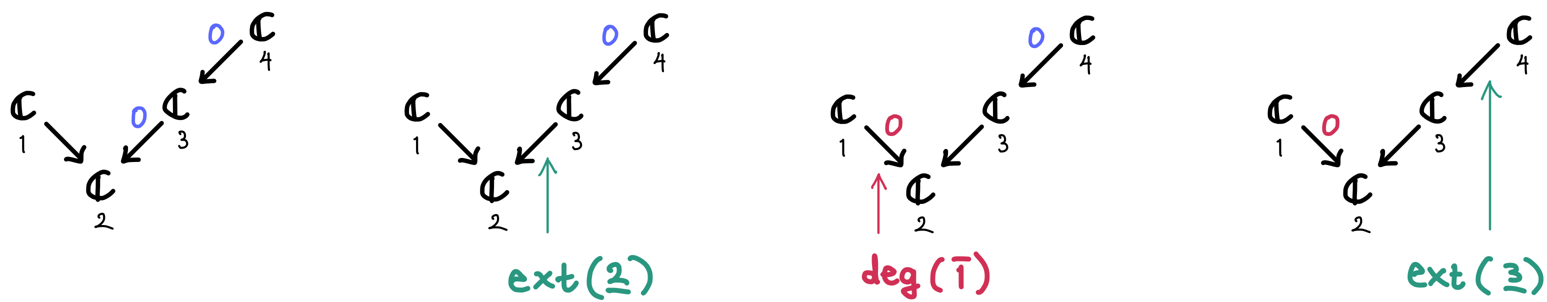
ext$(\underline{2})$ $\qquad$ deg$(\bar{1})$ $\qquad$ ext$(\underline{3})$

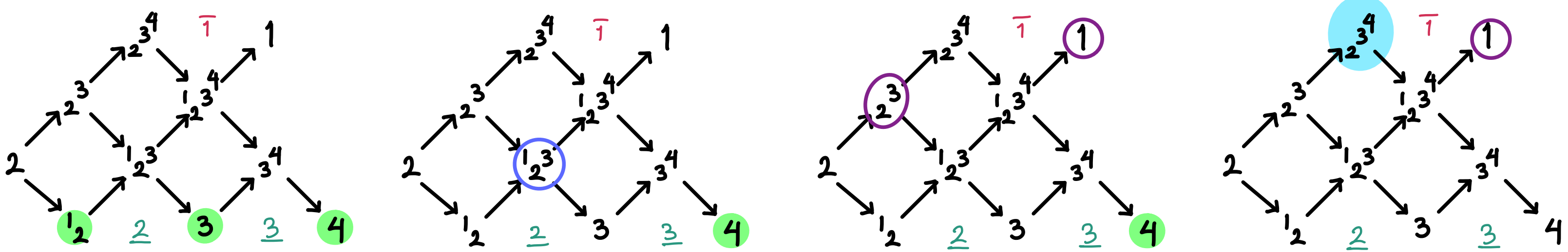
Take union of all 7 indecomposable summands to get m.a.r rep $\eta^r(213)$

$\eta^r : S_{n+1} \xrightarrow{\text{"union" of representations}} \mathrm{mar}(Q)$ via "rectangles with missing corners"

E.g. $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$ $\quad w(1) = \underline{2} \quad w(2) = \bar{1} \quad w(3) = \underline{3}$

$\lambda_0^r = {}^{1}_{2} \oplus 3 \oplus 4 \qquad \lambda_1^r = {}^{1}{}_{2}{}^{3} \oplus 4 \qquad \lambda_2^r = 1 \oplus {}_{2}{}^{3} \oplus 4 \qquad \lambda_3^r = 1 \oplus {}_{2}{}^{3^{4}}$





Take union of all 7 circled indecomposables to get m.a.r rep $\eta^r(213)$

THANK

YOU!

Stability function

Let $F : \{ \underbrace{\text{line segments } \gamma(i,j)}_{\mathcal{E}} \} \longrightarrow \operatorname{Ind} Q$

$$G := F^{-1} : \operatorname{Ind} Q \longrightarrow \mathcal{E}$$

Let $\operatorname{vec} : \mathcal{E} \longrightarrow \mathbb{C}$

$$\gamma(i,j) \longmapsto r e^{i\theta}$$

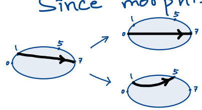
Stability function

$$\phi(M) = \frac{1}{\pi}\theta(M) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Prop All $M \in \operatorname{Ind} Q$ are ϕ-stable.

Proof If $L_{\neq 0}$ is a proper subrepresentation of M, the inclusion $L \hookrightarrow M$ is a nonzero morphism in $\operatorname{Hom}(L, M)$.

Since morphism γ_L to γ_M is a sequence of counterclock pivots like



, slope of $\operatorname{vec} \circ G(L) <$ slope of $\operatorname{vec} \circ G(M)$, so $\phi(L) < \phi(M)$. $\square$

- Let $D^b(\operatorname{rep} Q)$ be the derived category of bounded complexes in $\operatorname{rep} Q$.
- In type A, the indecomposable objects are of the form $M[i]$, $i \in \mathbb{Z}$.
- The line segment for $M[i]$ is $\begin{cases} \text{the same as } M = M[0] & \text{if } i \text{ is even} \\ \text{opposite orientation of } M = M[0] & \text{if } i \text{ is odd} \end{cases}$
- Stability function $\phi(M[j]) = \phi(M) + j$

extra