Cluster algebraic interpretation of infinite friezes

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Finite frieze patterns

Definition

A (Conway-Coxeter) **frieze pattern** is an array such that:

1. the top row is a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

   satisfies the rule $ad - bc = 1$.

Example (a **finite** integer frieze)

$$\begin{array}{lccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\text{Row 2} & \cdots & & \color{blue}{\mathbf{3}} & & \color{red}{\mathbf{1}} & & \color{green}{\mathbf{2}} & & \color{purple}{\mathbf{2}} & & \color{magenta}{\mathbf{1}} & & 3 & & 1 & \\
 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
 & \cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

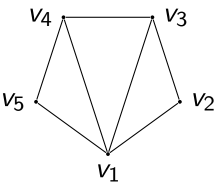
Note: every frieze pattern is completely determined by the 2nd row.

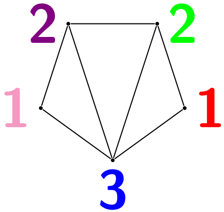
Conway and Coxeter (1970s)

Theorem

Finite frieze patterns $\longleftrightarrow$ triangulations of polygons

		1		1		1		1		1		1		1		$\cdots$
Row 2	$\cdots$		3		1		2		2		1		3		1	
		2		2		1		3		1		2		2		$\cdots$
	$\cdots$		1		1		1		1		1		1		1	





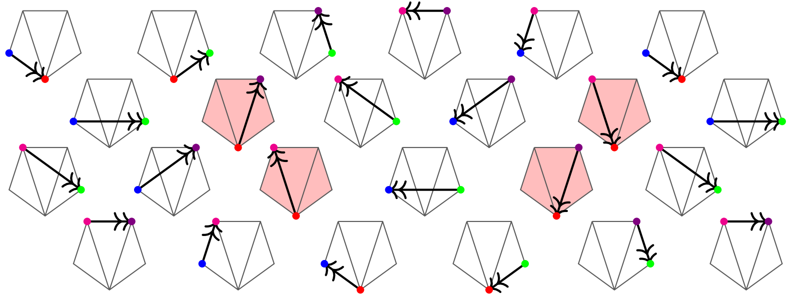
Broline, Crowe, and Isaacs (BCI, 1970s)

Theorem

Entries of a finite frieze pattern $\longleftrightarrow$ edges between two vertices.

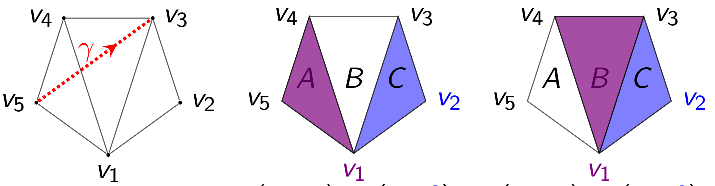
		1		1		1		1		1		1		⋯
Row 2	⋯		3		**1**		2		2		**1**		3	
		2		2		**1**		3		**1**		2		⋯
	⋯		1		1		1		1		1		1	

⋯



⋯

Broline, Crowe, and Isaacs (BCI, 1970s)



T and γ $\quad (t_1, t_2) = (A, C) \quad (t_1, t_2) = (B, C)$

Definition (BCI tuple)

Let $R_1, R_2, \ldots, R_r$ be the boundary vertices to the right of γ. A **BCI tuple** for γ is an r-tuple $(t_1, \ldots, t_r)$ such that:

(B1) the i-th entry t_i is a triangle of T having R_i as a vertex. (We say that the vertex R_i is matched to the triangle in the i-th entry of the tuple).

(B2) the entries are pairwise distinct.

Cluster algebras (Fomin and Zelevinsky, 2000)

A **cluster algebra** is a commutative ring with a distinguished set of generators, called **cluster variables**.

Cluster algebras from surfaces (Fomin, Shapiro, and Thurston, 2006, etc.)

- Fix a marked surface: a Riemann surface S + marked points.
- Points are either on the boundary of S or in the interior (called punctures).
- The cluster variables $\longleftrightarrow$ arcs with no self-intersection.

Remark

- *A cluster algebra of type A arises from a polygon.*
- *A cluster algebra of type D arises from a punctured polygon.*

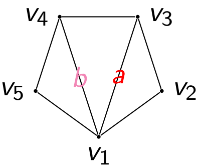
Caldero-Chapoton (2006)

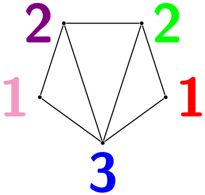
Theorem

The cluster variables of a cluster algebra from a triangulated polygon (type A) form a finite frieze pattern.

$$
\begin{array}{ccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & \\
\cdots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

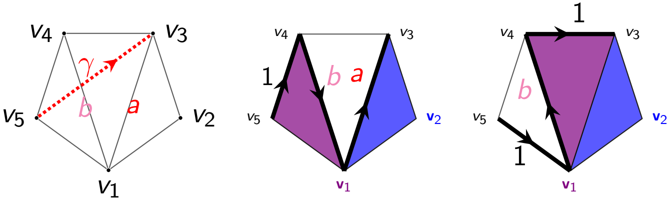
(Example: type A_2)





- Remark: When the variables are specialized to 1, we recover the integer frieze pattern.

BCI tuples to a cluster variable



T and γ $\qquad$ $1\,b^{-1}\,a$ $\qquad$ $1\,b^{-1}\,1$

Definition (Carroll-Price, 2003 and others)

A **BCI trail** w for $(t_1, \ldots, t_r)$ is a walk from the beginning to the ending point of γ along T such that:

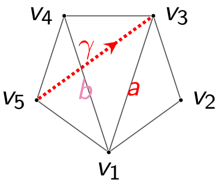
(TR 1) the triangles $t_1, \ldots, t_r$ are to the right of w,

(TR 2) the other triangles are to the left of w.

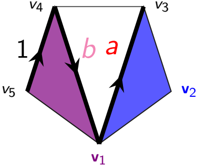
Proposition (Carroll-Price, 2003 and others)

There is a lattice-preserving bijection between the BCI tuples and T-paths (of Schiffler-Thomas, 2006-2007).

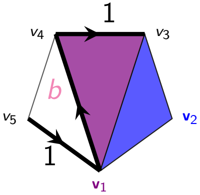
BCI tuples to a cluster variable



T and γ



$1\, b^{-1}\, a$



$1\, b^{-1}\, 1$

$$x_\gamma = \frac{a}{b} + \frac{1}{b}$$

Theorem (Carroll-Price, Schiffler-Thomas, and others)

1. BCI-trail formula: the Laurent polynomial expansion corresponding to γ written in the variables of T is

$$x_\gamma = \sum_w \frac{\prod \textit{odd steps of } w}{\prod \textit{even steps of } w}$$

where the sum is over all BCI-trails w for γ.

2. Starting from the minimal BCI-tuple for γ, we get all the BCI-tuples by "toggling" to a triangle closer to the starting point

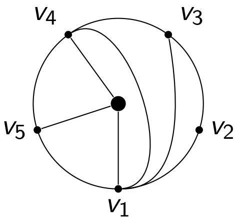
An infinite frieze pattern

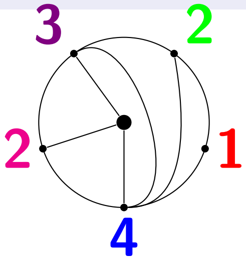
```
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
     4   1   2   3   2   4   1   2   3   2   4   1   2   3   2
       3   1   5   5   7   3   1   5   5   7   3   1   5   5   7
         2   2   8  17   5   2   2   8  17   5   2   2   8  17
Level 1    3   3  27  12   3   3   3  27  12   3   3   3  27  12
           ----------------------------------------------------
             4  10  19   7   4   4  10  19   7   4   4  10  19
              13   7  11   9   5  13   7  11   9   5  13   7  11
                 9   4  14  11  16   9   4  14  11  16   9   4
Level 2            5   5  17  35  11   5   5  17  35  11   5   5
                     6   6  54  24   6   6   6  54  24   6   6
                     -------------------------------------------
                       7  19  37  13   7   7  19  37  13   7   7
Level 3                 22  13  20  15   8  22  13  20  15   8
                          15   7  23  17  25  15   7  23  17  25
                             8   8  26  53  17   8   8  26  53
                               9   9  81  36   9   9   9  81  36
                               ---------------------------------
                                10  28  55  19  10  10  28  55
Level 4                           31  19  29  21  11  31  19  29
                                    21  10  32  23  34  21  10
```

Infinite frieze patterns

Theorem (Baur, Fellner, Parsons, and Tschabold, 2015-2016)

Any **infinite** *frieze can be constructed from a triangulation of a punctured disk or an annulus/ infinite strip.*





1 1 1 1 1 1 1 1 1 1 1

4 1 2 3 2 4 1 2 3 2

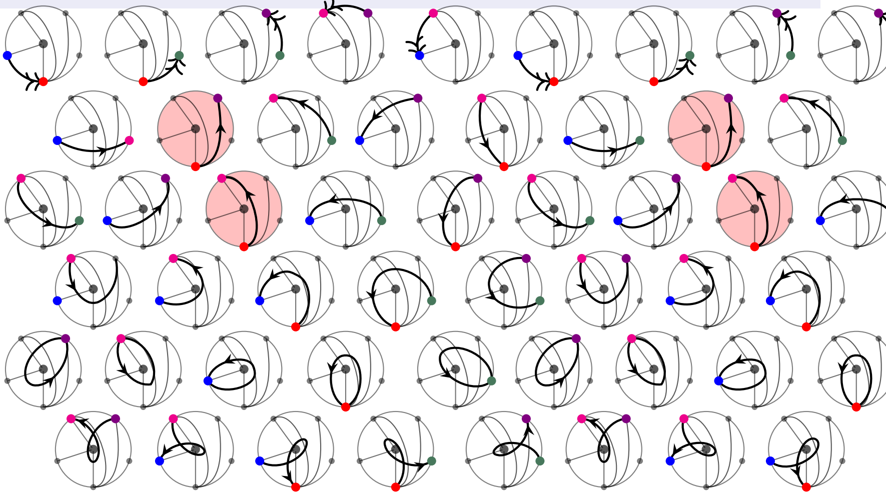
3 1 5 5 7 3 1 5 5 7

2 2 8 17 5 2 2 8 17 5

3 3 27 12 3 3 3 27 12

Theorem (G., Musiker, Vogel)

We construct an infinite frieze pattern of Laurent polynomials corresponding to arcs (allowing self-intersections) between the boundary vertices of a punctured disk or annulus.



- Convention: the boundary is to the right of the curve.

- Remark: When the variables are specialized to 1, we recover the integer frieze pattern.

1 1 1 1 1 1 1 1 1 1 1

4 **1** 2 3 2 4 **1** 2 3 2 4

3 **1** 5 5 7 3 **1** 5 5 7

2 2 8 17 5 2 2 8 17 5

3 3 27 12 3 3 3 27 12

- -

4 10 19 7 4 4 10 19 7

13 7 11 9 5 13 7 11

9 4 14 11 16 9 4 14

5 5 17 35 11 5 5

6 6 54 24 6 6 6

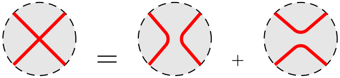
- - - - - - - - - - - - - - - - - -

7 19 37 13 7 7

Theorem (G., Musiker, Vogel)

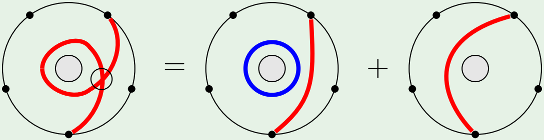
We construct an infinite frieze pattern of Laurent polynomials corresponding to arcs (allowing self-intersections) between the boundary vertices of a punctured disk or annulus.

Proof: The self-intersecting arcs correspond to elements of the algebra via skein relation



due to Musiker, Schiffler, and Williams (2011), and others.

Example (Example of resolving a self-crossing)

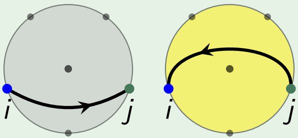


Complementary arcs

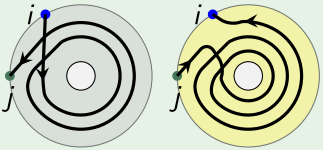
Definition (complementary arc)

Let $i < j$ and let γ_k be the arc from i to j with $k-1$ self-crossings. The **complementary arc** γ_k^C of γ_k is the arc from j to i with $k-1$ self-crossings.

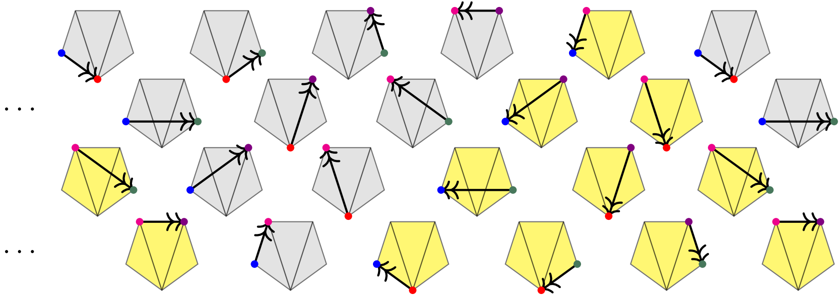
Example (γ_1 and γ_1^C)



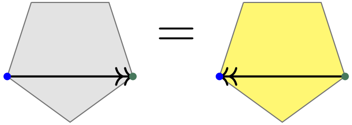
Example (γ_3 and γ_3^C)

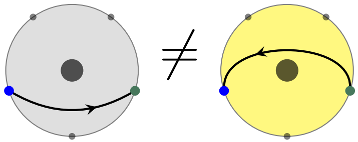


Glide symmetry for finite friezes

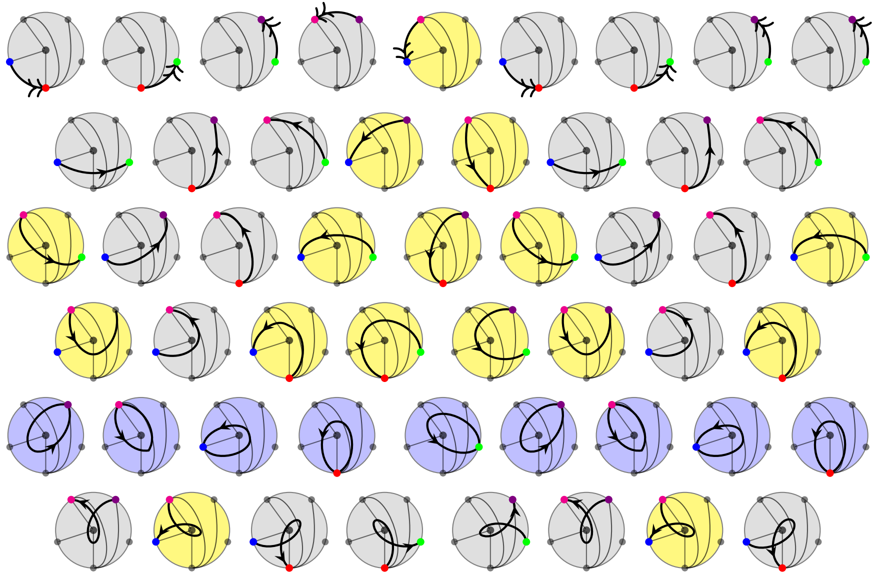


In a polygon vs a punctured disk/annulus





Complementary arcs in infinite friezes



. . .

Progression formulas

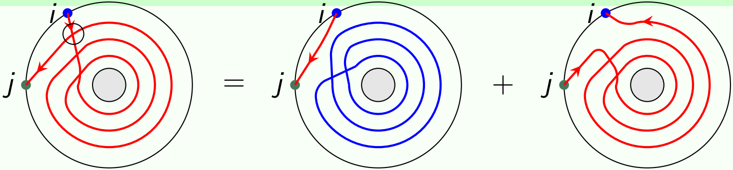
Theorem (G., Musiker, and Vogel)

Let γ_1 be an arc starting and finishing at vertices i and j. For $k = 1, 2, \dots$ and $1 \le m \le k-1$, we have

$$x(\gamma_k) = x(\gamma_m)x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}), \text{ where:}$$

- *for $r \ge 0$, γ^C_{-r} is the curve γ_{r+1} with a kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$, and*
- *a **bracelet** $Brac_k$ is obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, creating $(k-1)$ self-crossings.*

$$x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma^C_3) \text{ for } k = 4,\ m = 1$$



Arithmetic progressions in frieze patterns from punctured disks (Tschabold)

```
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
  4   1   2   3   2   4   1   2   3   2   4   1   2   3   2
    3   1   5   5   7   3   1   5   5   7   3   1   5   5   7
      2   2   8  17   5   2   2   8  17   5   2   2   8  17
        3   3  27  12   3   3   3  27  12   3   3   3  27  12
        -------------------------------------------------------
          4  10  19   7   4   4  10  19   7   4   4  10  19
           13   7  11   9   5  13   7  11   9   5  13   7  11
              9   4  14  11  16   9   4  14  11  16   9   4
                5   5  17  35  11   5   5  17  35  11   5   5
                  6   6  54  24   6   6   6  54  24   6   6
                  -----------------------------------------
                    7  19  37  13   7   7  19  37  13   7   7
                     22  13  20  15   8  22  13  20  15   8
                       15   7  23  17  25  15   7  23  17  25
                          8   8  26  53  17   8   8  26  53
                            9   9  81  36   9   9   9  81  36
                            ---------------------------------
                             10  28  55  19  10  10  28  55
                               31  19  29  21  11  31  19  29
```

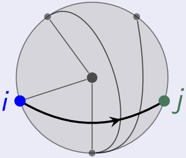
Geometric interpretation of the arithmetic progression

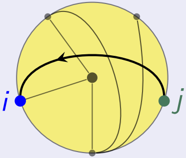
Proposition (G., Musiker, Vogel)

The arc from vertex blue to vertex green with k self-intersections

$$=$$

the arc from vertex blue to vertex green with $k-1$ self-intersections

$+$ 

$+$ 

Proof: Progression formulas and induction.

Constant **growth factor** across rows (Baur, Fellner, Parsons, Tschabold)

$$
\begin{array}{l|cccccc}
 & -1 & -1 & -1 & -1 & -1 & -1 \\
s_0 = 2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 2 & 6 & 1 & 2 & 6 \\
s_1 = 3 & \mathbf{1} & \mathbf{11} & \mathbf{5} & \mathbf{1} & \mathbf{11} & \mathbf{5} \\
\hline
 & 5 & 9 & 4 & 5 & 9 & 4 \\
 & 4 & 7 & 19 & 4 & 7 & 19 \\
s_2 = 7 & \mathbf{3} & \mathbf{33} & \mathbf{15} & \mathbf{3} & \mathbf{33} & \mathbf{15} \\
\hline
 & 14 & 26 & 11 & 14 & 26 & 11 \\
 & 11 & 19 & 51 & 11 & 19 & 51 \\
s_3 = 18 & \mathbf{8} & \mathbf{88} & \mathbf{40} & \mathbf{8} & \mathbf{88} & \mathbf{40} \\
\hline
 & 37 & 69 & 29 & 37 & 69 & 29 \\
 & 29 & 50 & 134 & 29 & 50 & 134 \\
 & & & \ddots & & & \ddots
\end{array}
$$

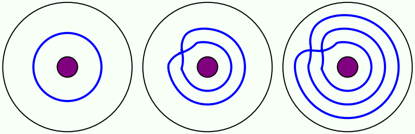
In punctured disk case, this growth factor is always 2.

```
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
  4   1   2   3   2   4   1   2   3   2   4   1   2   3   2   4   1   2   3
    3   1   5   5   7   3   1   5   5   7   3   1   5   5   7   3   1   5
      2   2   8  17   5   2   2   8  17   5   2   2   8  17   5   2   2   8
        3   3  27  12   3   3   3  27  12   3   3   3  27  12   3   3   3   2
        ------------------------------------------------------------------
          4  10  19   7   4   4  10  19   7   4   4  10  19   7   4   4  10
            13   7  11   9   5  13   7  11   9   5  13   7  11   9   5  13
               9   4  14  11  16   9   4  14  11  16   9   4  14  11  16   9
                 5   5  17  35  11   5   5  17  35  11   5   5  17  35  11
                   6   6  54  24   6   6   6  54  24   6   6   6  54  24   6
                   -------------------------------------------------------
                     7  19  37  13   7   7  19  37  13   7   7  19  37  13
                      22  13  20  15   8  22  13  20  15   8  22  13  20  15
                        15   7  23  17  25  15   7  23  17  25  15   7  23   1
                           8   8  26  53  17   8   8  26  53  17   8   8  26
                             9   9  81  36   9   9   9  81  36   9   9   9   8
                             --------------------------------------------
                              10  28  55  19  10  10  28  55  19  10  10  28
                                31  19  29  21  11  31  19  29  21  11  31   1
                                  21  10  32  23  34  21  10  32  23  34  21
                                    11  11  35  71  23  11  11  35  71  23   1
                                      12  12 108  48  12  12  12 108  48  12
                                      ----------------------------------
                                        13  37  73  25  13  13  37  73  25   1
                                          40  25  38  27  14  40  25  38
```

Geometric interpretation of the growth factor

The "jump" between frieze level k and $k+1$ correspond to the bracelet which crosses itself $k-1$ times.

Bracelets with 0, 1, and 2 self-crossings



Definition

Define the **normalized Chebyshev polynomial** by

$$T_0(x) = 2, T_1(x) = x, \text{ and}$$

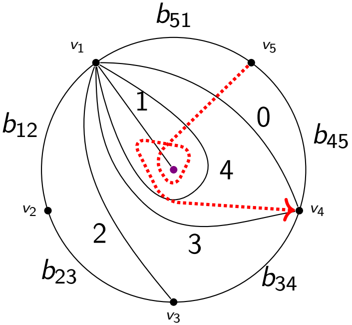
the recurrence relation

$$T_k(x) = x\, T_{k-1}(x) - T_{k-2}(x).$$

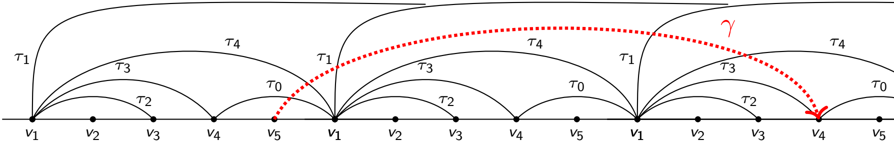
For punctured disk, every bracelet corresponds to the integer 2.

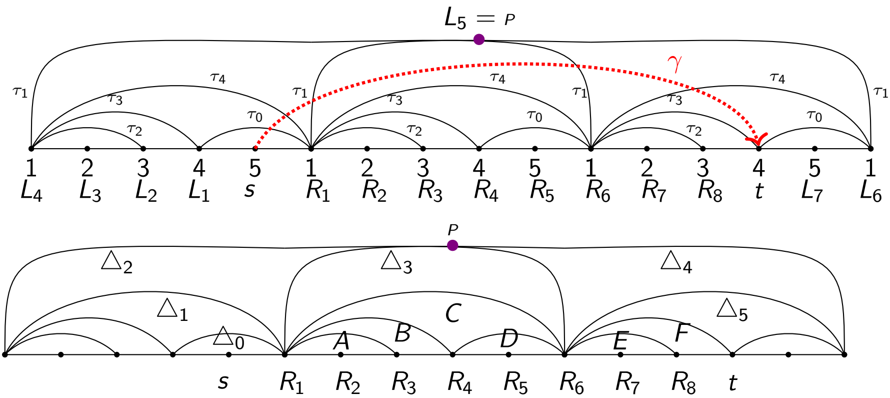
Thank you

From ideal triangulation T to its polygon cover



$\rightsquigarrow$




$L_5 = P$
γ
τ_1 τ_4 τ_3 τ_2 τ_0
1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1
L_4 L_3 L_2 L_1 s R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 t L_7 L_6
P
$\triangle_2$ $\triangle_3$ $\triangle_4$ $\triangle_1$ $\triangle_0$ $\triangle_5$
A B C D E F
s R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 t

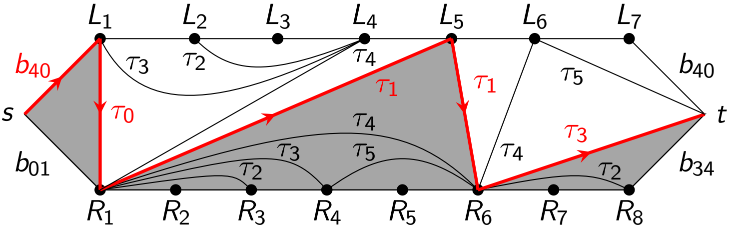
$$
\begin{tikzcd}
& (\Delta_3, A, B, C, D, \Delta_5, E, F) & & & \\
(\Delta_3, A, B, C, D, \Delta_4, E, F) \arrow[ur, "R_6"] & & (\Delta_2, A, B, C, D, \Delta_5, E, F) \arrow[ul, "R_1"'] & & \\
& (\Delta_2, A, B, C, D, \Delta_4, E, F) \arrow[ul, "R_1"] \arrow[ur, "R_6"] & & (\Delta_1, A, B, C, D, \Delta_5, E, F) \arrow[ul, "R_1"'] & \\
(\Delta_2, A, B, C, D, \Delta_3, E, F) \arrow[ur, "R_6"] & & (\Delta_1, A, B, C, D, \Delta_4, E, F) \arrow[ul, "R_1"] \arrow[ur, "R_6"] & & (\Delta_0, A, B, C, D, \Delta_5, E, F) \arrow[ul, "R_1"'] \\
& (\Delta_1, A, B, C, D, \Delta_3, E, F) \arrow[ul, "R_1"] \arrow[ur, "R_6"] & & (\Delta_0, A, B, C, D, \Delta_4, E, F) \arrow[ul, "R_1"] \arrow[ur, "R_6"'] & \\
& & (\Delta_0, A, B, C, D, \Delta_3, E, F) \arrow[ul, "R_1"] \arrow[ur, "R_6"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & 3 & & 5 \\
& 2 \arrow[ur] & & 4 \arrow[ul] \arrow[ur] & \\
1 \arrow[ur] & & & &
\end{tikzcd}
$$

The 11 BCI tuples correspond to the 11 terms of the expansion of x_γ:

$$x_\gamma = \frac{\mathbf{x_0 x_1 x_4} + 2x_1x_3x_4 + 2x_0^2 + 4x_0x_3 + 2x_3^2}{x_0x_1x_4}$$

For example, from the minimal BCI tuple $b = (\Delta_0, A, B, C, D, \Delta_3, E, F)$, we get a BCI trail $(b_{40}, \tau_5, \tau_1, \tau_1, \tau_3)$.



$$\text{with weight } b_{40}\, x_0^{-1}\, x_1\, x_1^{-1}\, x_3 = \frac{1\, x_1\, x_3}{x_0\, x_1}$$