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A new Motzkin object from box-ball systems

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Abstract

BBS-good tableaux are a class of tableaux arising from box-ball systems. We show that the BBS-good tableaux of size n are counted by the n-th Motzkin number.

Box-ball systems

Define the discrete dynamical **box-ball system** (BBS) of $w \in S_n$ as follows.

- Label n balls by the one-line notation of w and place them in an infinite empty strip.
- Move the ball labeled 1 to the first empty box to its right and vacate the box from which it moved. Repeat in order from 2 to n.
- After all n balls have moved, we have completed a **box-ball move** and we advance time.

This version of the box-ball system was introduced by Daisuke Takahashi in [6] and is an extension of the box-ball system first invented by Takahashi and Junkichi Satsuma in [7]. For example, given $452361 \in S_6$, the intermediate steps between $t = 0$ and $t = 1$ are

$t=0$	...		4	5	2	3	6	1					...
	...		4	5	2	3	6		1				...
	...		4	5		3	6	2	1				...
	...		4	5			6	2	1	3			...
	...			5	4		6	2	1	3			...
	...				4	5	6	2	1	3			...
$t=1$	...				4	5		2	1	3	6		...

Evolving the system through several box-ball moves we get

$t=0$	...		4	5	2	3	6	1																...
$t=1$	...				4	5		2	1	3	6													...
$t=2$	...						4	5	2			1	3	6										...
$t=3$	...								4	2	5				1	3	6							...
$t=4$	...									4		2	5					1	3	6				...
$t=5$	...										4			2	5						1	3	6		...

A **soliton** is a maximal consecutive increasing sequence of balls that is preserved by all future BBS moves. A BBS configuration is said to be in **steady state** when every ball is contained in a soliton.

Theorem 1 (Takahashi [6]): After a finite number of BBS moves, a box-ball system containing a configuration $w \in S_n$ will reach a steady state, decomposing into solitons whose sizes are weakly increasing from left to right, that is, forming an integer partition of n.

The **soliton decomposition** of a box-ball system is the tableau in which the first row is the rightmost soliton, the second row is the next soliton to its left, and so on. We call a permutation **BBS-good**, or just **good**, if its SD is a standard Young tableau. For instance,

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, \quad \text{and } 452361 \text{ is good}$$

Robinson-Schensted (RS) correspondence

We will use the Robinson–Schensted insertion algorithm to study the box-ball system. The Robinson–Schensted (RS) insertion algorithm is a well-studied bijection between permutations $w \in S_n$ and pairs of standard Young tableaux $(\mathrm{P}(w), \mathrm{Q}(w))$ of the same shape of size n [5].

$$\text{For instance } 452361 \xrightarrow{\mathrm{RS}} \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \right)$$

The first tableau is called the **insertion tableau** and the second is called the **recording tableau**. It turns out that the recording tableau largely controls box-ball dynamics.

Theorem 2 (DGG+ [2]): w is good if and only if the shape of $\mathrm{SD}(w)$ is the same as the shape of $\mathrm{Q}(w)$.

Theorem 3 (CFG+ [1]): If $\mathrm{Q}(w) = \mathrm{Q}(v)$, then $\mathrm{SD}(w)$ and $\mathrm{SD}(v)$ have the same shape.

In light of these theorems, we define a standard Young tableau T to be **good** when w is good for any w such that $\mathrm{Q}(w) = T$.

Motzkin numbers

The n-**th Motzkin number**, denoted M_n is defined by the two term recursion

$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-k-2}$$

with $M_0 = M_1 = 1$.

Theorem 4: The good tableaux of size n are counted by the n-th Motzkin number.

To prove this theorem, we define operations on good tableaux mirroring each component of the Motzkin recursion, that is, an operation that takes a size $n-1$ tableau and returns a size n tableau, an operation that takes a size $n-k-2$ tableau and returns a size $n-k$ tableau, and an operation that takes a size k and a size $n-k$ tableaux and returns a size n tableau.

A notion of tableau multiplication

To try and recreate the multiplication of the M_k and M_{n-k-2} terms in the Motzkin recursion, we introduce a binary operation for tableaux.

Let T_1, T_2 be standard Young tableaux of size n_1, n_2. Let $\overline{T_1}$ be a tableau of the same shape as T_1 but with each value j replaced with $j + n_1$. Define $T_1 \widetilde{\times} T_2$ as the tableau resulting from placing $\overline{T_1}$ below T_2 and "flushing up" the entries.

$$\text{If } T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \; T_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \text{ then } \overline{T_1} = \begin{array}{|c|c|c|}\hline 3 & 4 & 7 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} \text{ and } T_1 \widetilde{\times} T_2 = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Proposition 5: $T_1 \widetilde{\times} T_2$ is good if and only if T_1 and T_2 are good.

Column bump and row wrap

Let T be a standard tableau of size n. Construct the **column bump** of T, denoted $\mathrm{bump}(T)$, to be $T \widetilde{\times} \boxed{1}$. Construct the **row wrap**, denoted $\mathrm{wrap}(T)$, by increasing every element of T by 1 to get T', and then replacing the first row of T' with a copy of the first row but with 1 at the beginning and $n+2$ at the end.

$$\text{If } T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}, \text{ then } \mathrm{bump}(T) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}, \quad \mathrm{wrap}(T) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 8 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

Proposition 6: For a standard Young tableau T, the following are equivalent:

- T is good.
- $\mathrm{bump}(T)$ is good.
- $\mathrm{wrap}(T)$ is good.

A Motzkin class of tableaux

There is a unique decomposition of sorts for tableaux of the form $T \widetilde{\times} \mathrm{wrap}(Q)$ for good tableaux T, Q as shown in the following proposition.

Proposition 7: Suppose T_1, T_2, Q_1, Q_2 are good tableaux such that

$$T = T_1 \widetilde{\times} \mathrm{wrap}(Q_1) = T_2 \widetilde{\times} \mathrm{wrap}(Q_2)$$

Then $T_1 = T_2$ and $Q_1 = Q_2$.

In light of this proposition, we now recursively define the sets of tableaux K_n as follows. Let $K_0 = \{\emptyset\}$ and $K_1 = \left\{\boxed{1}\right\}$ where $\emptyset$ denotes the empty tableau. Then, for $n \geq 2$,

1. for each $Q \in K_{n-1}$, $\mathrm{bump}(Q) \in K_n$
2. for each pair of $Q_1 \in K_k$ and $Q_2 \in K_{n-k-2}$ for $0 \leq k \leq n-2$, $Q_1 \widetilde{\times} \mathrm{wrap}(Q_2) \in K_n$

No other tableaux are in K_n.

Theorem 8: For each $n \geq 0$, each $T \in K_n$ is good and $|K_n| = M_n$.

This theorem constitutes one direction of the main theorem, i.e. showing there is a class of good tableaux which are themselves a Motzkin object.

A simple characterization of goodness

We call a subsequence σ of w a k-**decreasing subsequence** when $\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \ldots \sqcup \sigma_k$ with each σ_i decreasing or empty. We define $\mathrm{d}_k(w)$ to be the length of a longest k-decreasing subsequence of w.

An index i is a **descent** of w if $w(i+1) < w(i)$. We define

$$\mathrm{D}(u) := \begin{cases} 0 & \text{if } u \text{ is empty} \\ 1 + |\{\text{descents of } u\}| & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{D}_k(w) := \max_{w = u_1 \sqcup \cdots \sqcup u_k} \sum_{j=1}^{k} \mathrm{D}(u_j)$$

The following combines results of Schensted [5], Greene [3], and Lewis et al. [4] to characterize good permutations.

Theorem 9: A permutation w is good if and only if $\mathrm{d}_k(w) = \mathrm{D}_k(w)$ for each $k \geq 1$.

Column-superstandard permutations

To exploit the previous characterization of goodness we introduce a class of permutations with relatively simple $\mathrm{d}_k, \mathrm{D}_k$ statistics. Given a standard Young tableau Q, define the **column-superstandard permutation** of Q as the RS inverse of $(\mathrm{CSS}(\mathrm{sh}\, Q), Q)$ where $\mathrm{CSS}(\mathrm{sh}\, Q)$ is the column-superstandard tableau of shape $\mathrm{sh}\, Q$ (see example below):

$$Q = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 & 11 \\ \hline 4 & 5 & 10 & 22 \\ \cline{1-4} 8 & 9 & 14 \\ \cline{1-3} 12 & 13 & 17 \\ \cline{1-3} 15 & 16 & 21 \\ \cline{1-3} 18 & 20 \\ \cline{1-2} 19 \\ \cline{1-1}\end{array} \quad \text{and} \quad \mathrm{CSS}(\mathrm{sh}\, Q) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 8 & 14 & 19 & 21 & 22 \\ \hline 2 & 9 & 15 & 20 \\ \cline{1-4} 3 & 10 & 16 \\ \cline{1-3} 4 & 11 & 17 \\ \cline{1-3} 5 & 12 & 18 \\ \cline{1-3} 6 & 13 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

so if $v = \mathrm{RS}^{-1}(\mathrm{CSS}(\mathrm{sh}\, Q)), Q)$ is the column-superstandard permutation of Q, then v is

$v(1)$	$v(2)$	$v(3)$	$v(4)$	$v(5)$	$v(6)$	$v(7)$	$v(8)$	$v(9)$	$v(10)$	$v(11)$	$v(12)$	$v(13)$	$v(14)$	$v(15)$	$v(16)$	$v(17)$	$v(18)$	$v(19)$	$v(20)$	$v(21)$	$v(22)$
7	(13)	(18)	6	(12)	(20)	(21)	5	(11)	(17)	(22)	4	(10)	(16)	3	9	(15)	2	1	8	(14)	(19)

Remark 10: The columns of $\mathrm{CSS}(\mathrm{sh}\, Q)$ appear as decreasing subsequences of v, where the positions of these subsequences are controlled in a simple way by Q.

Lemma 11: Let v be the column superstandard permutation of a standard tableau Q. Then $v(m)$ is the k-th entry from the bottom of the i-th column of $\mathrm{CSS}(\mathrm{sh}\, Q)$ if and only if $Q\,(k, i) = m$.

All good tableaux are in K_n

Column-superstandard permutations allow us to prove the following.

Lemma 12: Suppose Q is a good tableau of size $n \geq 2$. If Q_{n-1} is the tableau of size n formed by removing the entry n from Q, then Q_{n-1} is good.

Then, using induction, we can use this result to prove the other inclusion of the main theorem.

Theorem 13: Suppose Q is a good tableau of size n. Then $Q \in K_n$.

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