

Abstract

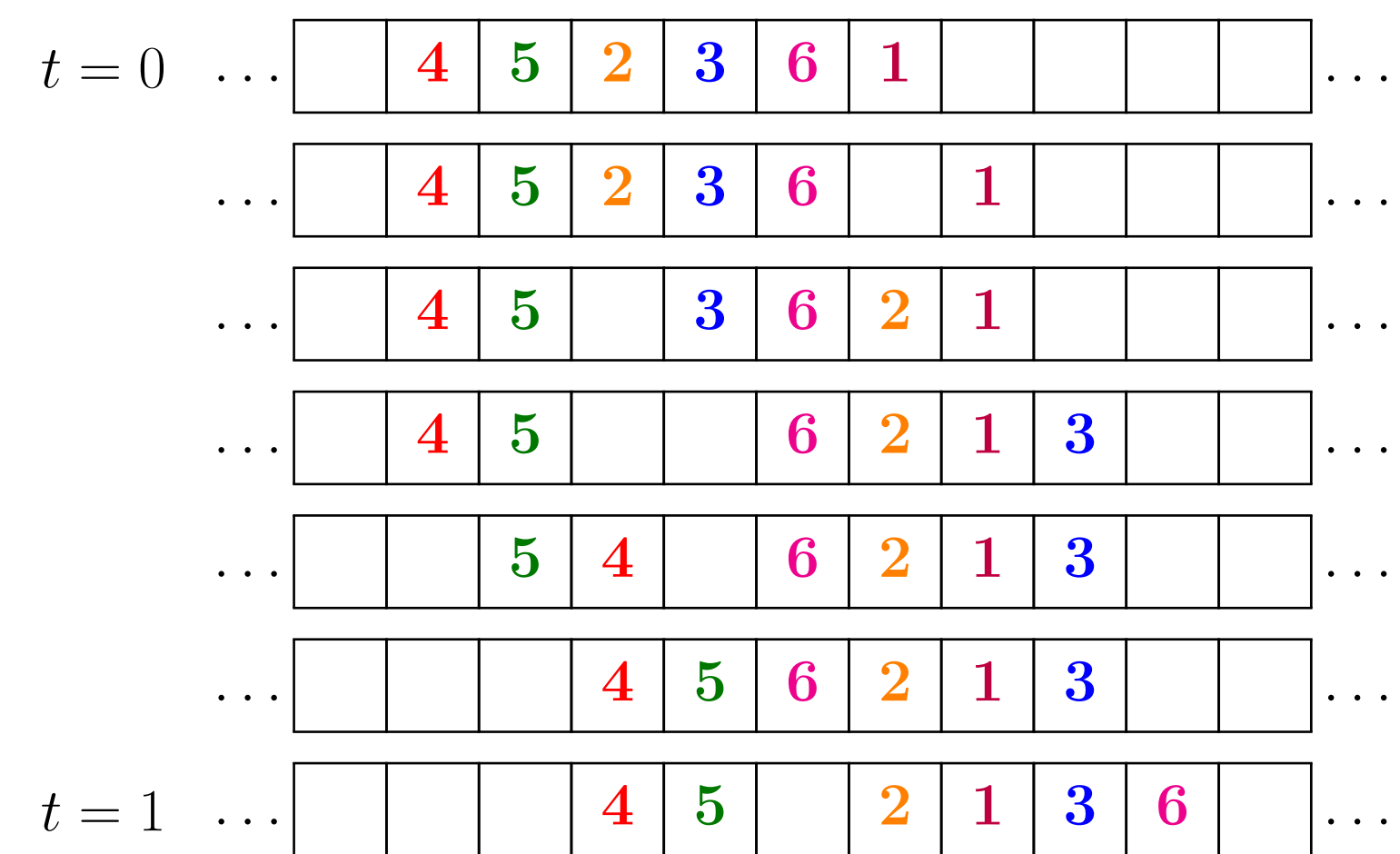
BBS-good tableaux are a class of tableaux arising from box-ball systems. We show that the BBS-good tableaux of size n are counted by the n -th Motzkin number.

Box-ball systems

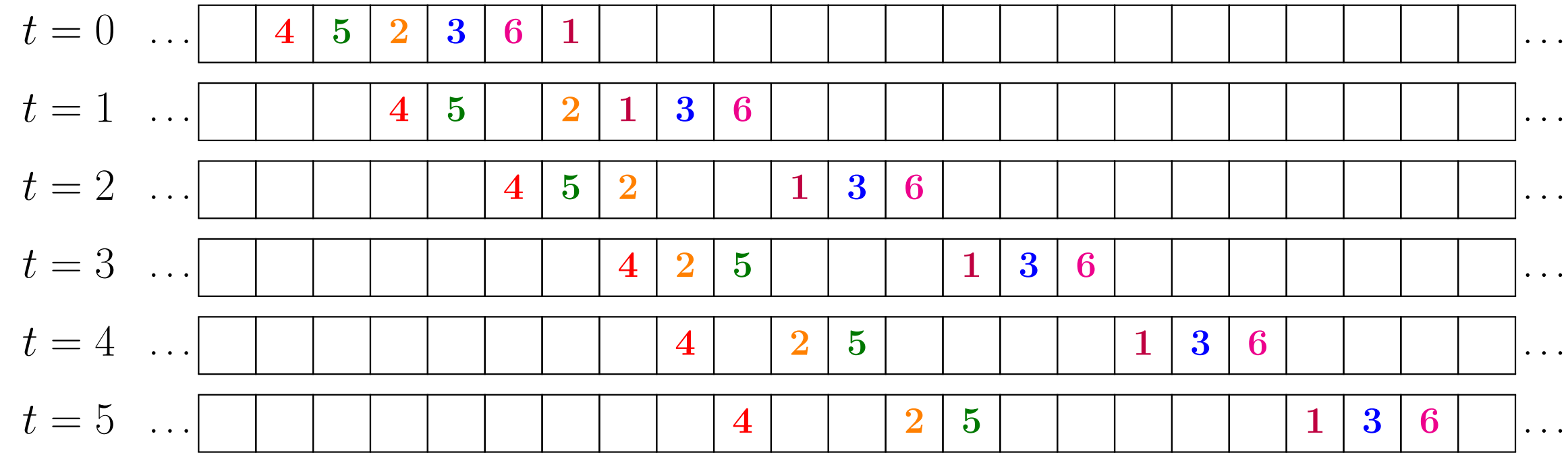
Define the discrete dynamical **box-ball system** (BBS) of $w \in S_n$ as follows.

- Label n balls by the one-line notation of w and place them in an infinite empty strip.
- Move the ball labeled 1 to the first empty box to its right and vacate the box from which it moved. Repeat in order from 2 to n .
- After all n balls have moved, we have completed a **box-ball move** and we advance time.

This version of the box-ball system was introduced by Daisuke Takahashi in [6] and is an extension of the box-ball system first invented by Takahashi and Junkichi Satsuma in [7]. For example, given $452361 \in S_6$, the intermediate steps between $t = 0$ and $t = 1$ are



Evolving the system through several box-ball moves we get



A **soliton** is a maximal consecutive increasing sequence of balls that is preserved by all future BBS moves. A BBS configuration is said to be in **steady state** when every ball is contained in a soliton.

Theorem 1 (Takahashi [6]): After a finite number of BBS moves, a box-ball system containing a configuration $w \in S_n$ will reach a steady state, decomposing into solitons whose sizes are weakly increasing from left to right, that is, forming an integer partition of n .

The **soliton decomposition** of a box-ball system is the tableau in which the first row is the rightmost soliton, the second row is the next soliton to its left, and so on. We call a permutation **BBS-good**, or just **good**, if its SD is a standard Young tableau. For instance,

$$\text{SD}(452361) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array}, \text{ and } 452361 \text{ is good}$$

Robinson-Schensted (RS) correspondence

We will use the Robinson-Schensted insertion algorithm to study the box-ball system. The Robinson-Schensted (RS) insertion algorithm is a well-studied bijection between permutations $w \in S_n$ and pairs of standard Young tableaux $(P(w), Q(w))$ of the same shape of size n [5].

$$\text{For instance } 452361 \xrightarrow{\text{RS}} \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array} \right)$$

The first tableau is called the **insertion tableau** and the second is called the **recording tableau**. It turns out that the recording tableau largely controls box-ball dynamics.

Theorem 2 (DGG+ [2]): w is good if and only if the shape of $\text{SD}(w)$ is the same as the shape of $Q(w)$.

Theorem 3 (CFG+ [1]): If $Q(w) = Q(v)$, then $\text{SD}(w)$ and $\text{SD}(v)$ have the same shape.

In light of these theorems, we define a standard Young tableau T to be **good** when w is good for any w such that $Q(w) = T$.

Motzkin numbers

The n -th **Motzkin number**, denoted M_n is defined by the two term recursion

$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-k-2}$$

with $M_0 = M_1 = 1$.

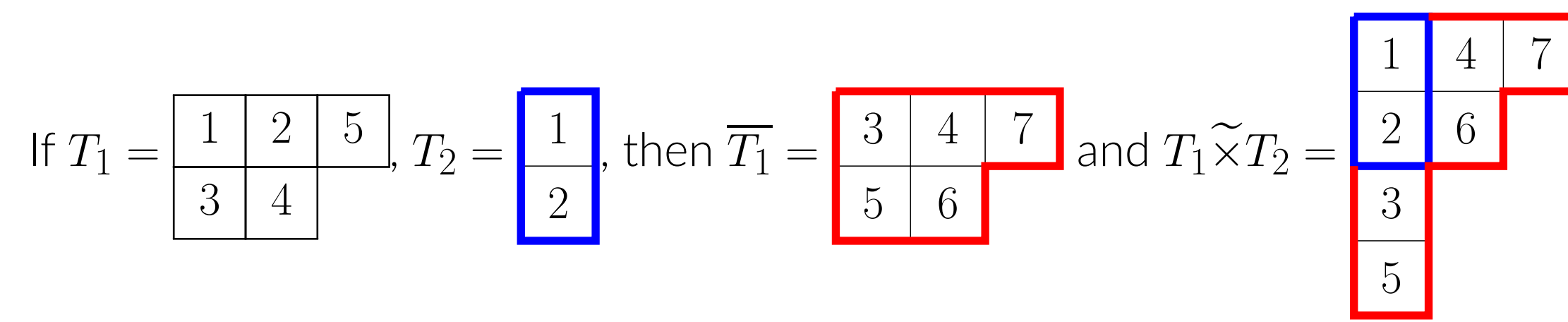
Theorem 4: The good tableaux of size n are counted by the n -th Motzkin number.

To prove this theorem, we define operations on good tableaux mirroring each component of the Motzkin recursion, that is, an operation that takes a size $n-1$ tableau and returns a size n tableau, an operation that takes a size $n-k-2$ tableau and returns a size $n-k$ tableau, and an operation that takes a size k and a size $n-k$ tableaux and returns a size n tableau.

A notion of tableau multiplication

To try and recreate the multiplication of the M_k and M_{n-k-2} terms in the Motzkin recursion, we introduce a binary operation for tableaux.

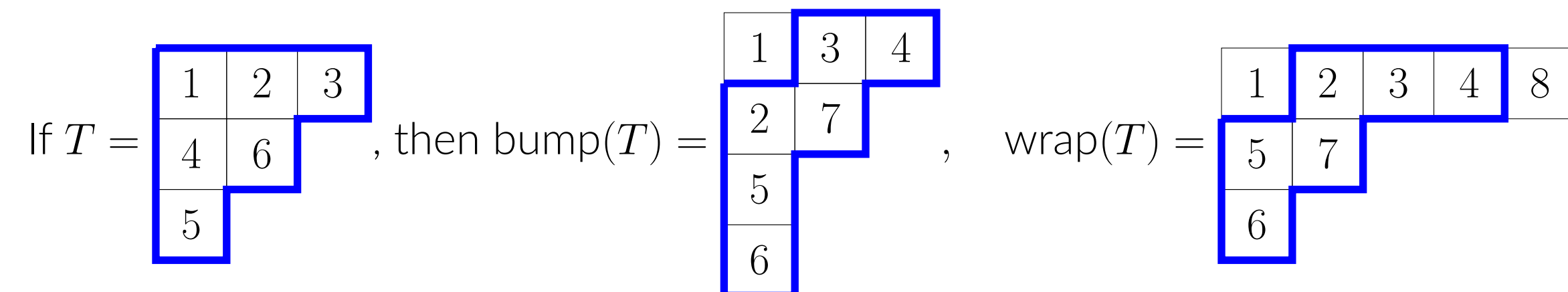
Let T_1, T_2 be standard Young tableaux of size n_1, n_2 . Let $\overline{T_1}$ be a tableau of the same shape as T_1 but with each value j replaced with $j+n_1$. Define $T_1 \tilde{\times} T_2$ as the tableau resulting from placing $\overline{T_1}$ below T_2 and "flushing up" the entries.



Proposition 5: $T_1 \tilde{\times} T_2$ is good if and only if T_1 and T_2 are good.

Column bump and row wrap

Let T be a standard tableau of size n . Construct the **column bump** of T , denoted $\text{bump}(T)$, to be $T \tilde{\times} \overline{[1]}$. Construct the **row wrap**, denoted $\text{wrap}(T)$, by increasing every element of T by 1 to get T' , and then replacing the first row of T' with a copy of the first row but with 1 at the beginning and $n+2$ at the end.



Proposition 6: For a standard Young tableau T , the following are equivalent:

- T is good.
- $\text{bump}(T)$ is good.
- $\text{wrap}(T)$ is good.

A Motzkin class of tableaux

There is a unique decomposition of sorts for tableaux of the form $T \tilde{\times} \text{wrap}(Q)$ for good tableaux T, Q as shown in the following proposition.

Proposition 7: Suppose T_1, T_2, Q_1, Q_2 are good tableaux such that

$$T = T_1 \tilde{\times} \text{wrap}(Q_1) = T_2 \tilde{\times} \text{wrap}(Q_2)$$

Then $T_1 = T_2$ and $Q_1 = Q_2$.

In light of this proposition, we now recursively define the sets of tableaux K_n as follows. Let $K_0 = \{\emptyset\}$ and $K_1 = \{\overline{[1]}\}$ where \emptyset denotes the empty tableau. Then, for $n \geq 2$,

- for each $Q \in K_{n-1}$, $\text{bump}(Q) \in K_n$
- for each pair of $Q_1 \in K_k$ and $Q_2 \in K_{n-k-2}$ for $0 \leq k \leq n-2$, $Q_1 \tilde{\times} \text{wrap}(Q_2) \in K_n$

No other tableaux are in K_n .

Theorem 8: For each $n \geq 0$, each $T \in K_n$ is good and $|K_n| = M_n$.

This theorem constitutes one direction of the main theorem, i.e. showing there is a class of good tableaux which are themselves a Motzkin object.

A simple characterization of goodness

We call a subsequence σ of w a **k -decreasing subsequence** when $\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \dots \sqcup \sigma_k$ with each σ_i decreasing or empty. We define $d_k(w)$ to be the length of a longest k -decreasing subsequence of w .

An index i is a **descent** of w if $w(i+1) < w(i)$. We define

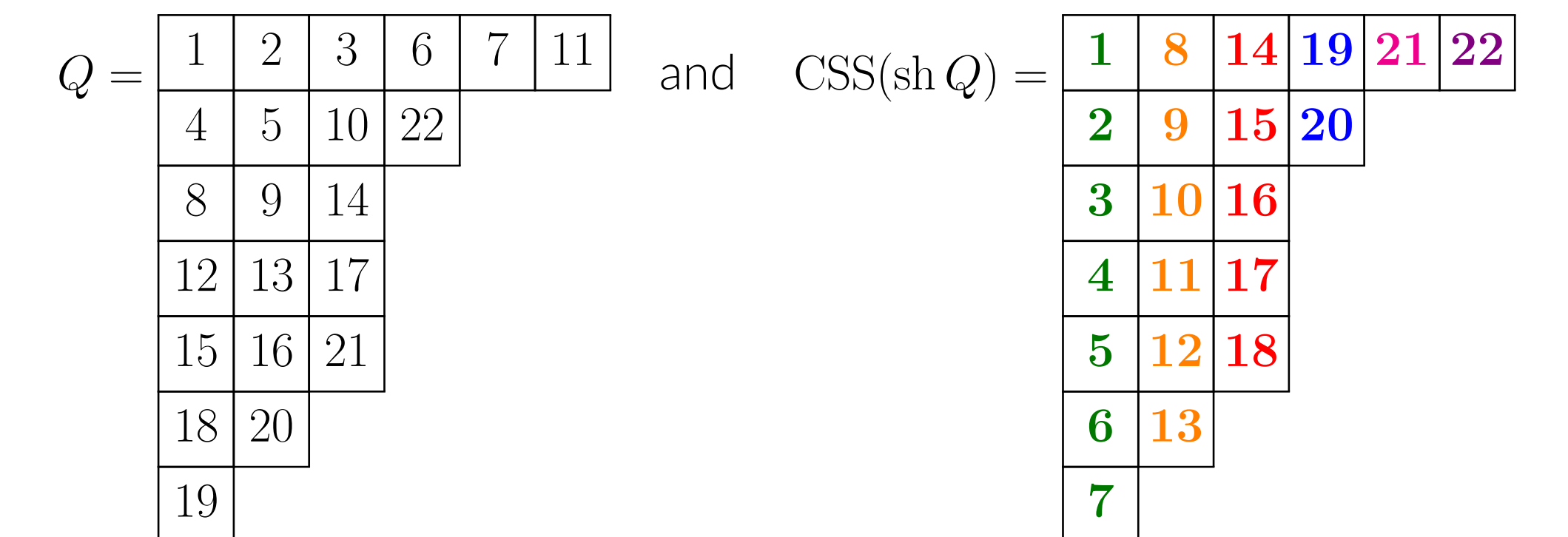
$$D(u) := \begin{cases} 0 & \text{if } u \text{ is empty} \\ 1 + \{\text{descents of } u\} & \text{otherwise} \end{cases} \quad \text{and} \quad D_k(w) := \max_{w=u_1 \sqcup \dots \sqcup u_k} \sum_{j=1}^k D(u_j)$$

The following combines results of Schensted [5], Greene [3], and Lewis et al. [4] to characterize good permutations.

Theorem 9: A permutation w is good if and only if $d_k(w) = D_k(w)$ for each $k \geq 1$.

Column-superstandard permutations

To exploit the previous characterization of goodness we introduce a class of permutations with relatively simple d_k, D_k statistics. Given a standard Young tableau Q , define the **column-superstandard permutation** of Q as the RS inverse of $(\text{CSS}(\text{sh } Q), Q)$ where $\text{CSS}(\text{sh } Q)$ is the column-superstandard tableau of shape $\text{sh } Q$ (see example below):



so if $v = \text{RS}^{-1}(\text{CSS}(\text{sh } Q), Q)$ is the column-superstandard permutation of Q , then v is

$$v(1) \ v(2) \ v(3) \ v(4) \ v(5) \ v(6) \ v(7) \ v(8) \ v(9) \ v(10) \ v(11) \ v(12) \ v(13) \ v(14) \ v(15) \ v(16) \ v(17) \ v(18) \ v(19) \ v(20) \ v(21) \ v(22) \\ 7 \ (13) \ (18) \ 6 \ (12) \ (20) \ (21) \ 5 \ (11) \ (17) \ (22) \ 4 \ (10) \ (16) \ 3 \ 9 \ (15) \ 2 \ 1 \ 8 \ (14) \ (19)$$

Remark 10: The columns of $\text{CSS}(\text{sh } Q)$ appear as decreasing subsequences of v , where the positions of these subsequences are controlled in a simple way by Q .

Lemma 11: Let v be the column superstandard permutation of a standard tableau Q . Then $v(m)$ is the k -th entry from the bottom of the i -th column of $\text{CSS}(\text{sh } Q)$ if and only if $Q(k, i) = m$.

All good tableaux are in K_n

Column-superstandard permutations allow us to prove the following.

Lemma 12: Suppose Q is a good tableau of size $n \geq 2$. If Q_{n-1} is the tableau of size n formed by removing the entry n from Q , then Q_{n-1} is good.

Then, using induction, we can use this result to prove the other inclusion of the main theorem.

Theorem 13: Suppose Q is a good tableau of size n . Then $Q \in K_n$.

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References

- Marisa Cofe, Olivia Fugikawa, Emily Gunawan, Madelyn Stewart, and David Zeng. Box-ball systems and RSK recording tableaux. Preprint (arXiv) 2209.09277, 2022.
- Ben Drucker, Eli Garcia, Emily Gunawan, Aubrey Rumbolt, and Rose Silver. RSK tableaux and box-ball systems. Preprint (arXiv) 2112.03780, 2021.
- Curtis Greene. An extension of Schensted's theorem. Advances in Math., 14:254-265, 1974.
- Joel Lewis, Hanbaek Lyu, Pavlo Pylyavskyy, and Arnab Sen. Scaling limit of soliton lengths in a multicolor box-ball system. Preprint (arXiv) 1911.04458, 2019.
- Craig Schensted. Longest increasing and decreasing subsequences. Canadian J. Math., 13:179-191, 1961.
- Daisuke Takahashi. On some soliton systems defined by using boxes and balls. In Proceedings of the international symposium on nonlinear theory and its applications (NOLTA'93), pages 555-558, 1993.
- Daisuke Takahashi and Junkichi Satsuma. A soliton cellular automaton. J. Phys. Soc. Japan, 59(10):3514-3519, 1990.