Box Ball Systems and Motzkin Objects

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Permutations

Let S_n be the set of all permutations on the numbers $\{1, 2, \ldots, n\}$. Today, we will express permutations in one-line notation.

Example

Here is the same permutation $w \in S_5$ expressed in two ways.

$$\textit{one-line: } w = 12354$$

$$\textit{two-line: } w = \left(\begin{smallmatrix} 12345 \\ 12354 \end{smallmatrix}\right)$$

Box-Ball Systems

A *box-ball system* is a dynamical system consisting of balls in an infinite strip. Balls take turns jumping to the first available box to their right, starting with the smallest-numbered ball.

Box-Ball Move Example

$t = 0$	…		4	5	2	3	6	1					…
	…		4	5	2	3	6		1				…
	…		4	5		3	6	2	1				…
	…		4	5			6	2	1	3			…
	…			5	4		6	2	1	3			…
	…				4	5	6	2	1	3			…
$t = 1$	…				4	5		2	1	3	6		…

Box-Ball System Example

$t = 0$	$\cdots$		4	5	2	3	6	1																$\cdots$
$t = 1$	$\cdots$				4	5		2	1	3	6													$\cdots$
$t = 2$	$\cdots$						4	5	2			1	3	6										$\cdots$
$t = 3$	$\cdots$								4	2	5				1	3	6							$\cdots$
$t = 4$	$\cdots$									4		2	5					1	3	6				$\cdots$
$t = 5$	$\cdots$										4			2	5						1	3	6		$\cdots$

Box-Ball Definitions

Eventually, any box-ball system decomposes into solitons.

Definition 1

A *soliton* is a consecutive sequence of balls, whose numbers are strictly increasing, that will remain together through all future box-ball moves.

Definition 2

When every ball is contained in a soliton, the system is in *steady state*. The system's *steady state time* or *sst* is the number of box-ball moves it takes to reach steady state.

Young Tableaux

Definition 3

A *Young tableau* is a diagram of boxes filled with positive integers such that the lengths of its rows and columns are weakly decreasing.

Example

2	1	1	4
7	3	2	
9	8	10	
1			

Young Tableaux (cont.)

Definition 4

A Young tableau is *standard* if the values in its rows and columns are increasing and each of the integers 1 through n appears exactly once.

Example

<table><tr><td>1</td><td>3</td><td>4</td><td>6</td></tr><tr><td>2</td><td>5</td><td>7</td><td></td></tr><tr><td>8</td><td></td><td></td><td></td></tr></table>

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<table><tr><td>1</td><td>2</td><td>4</td></tr><tr><td>3</td><td>5</td><td>9</td></tr><tr><td>6</td><td>8</td><td></td></tr><tr><td>7</td><td></td><td></td></tr></table>

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Soliton Decomposition

Definition 5

The *soliton decomposition* of a permutation is the tableau formed by taking its solitons to be the rows of a Young tableau, where the right-most soliton is the top row of the tableau.

Example

If we take $w = 452361$ *again, we can apply box-ball moves to find* $\mathrm{sst}(w)$ *as well as* $SD(w)$.

$t = 0$	...		4	5	2	3	6	1												...
$t = 1$	...				4	5		2	1	3	6									...
$t = 2$	...						4	5	2			1	3	6						...
$t = 3$	...								4	2	5				1	3	6			...

At $t = 3$, *the system is in steady state, thus* $\mathrm{sst}(w) = 3$ *and* $SD(w) =$

1	3	6
2	5	
4		

BBS-good permutations

Definition 6

A permutation w is *BBS-good*, or simply *good* if $\mathrm{SD}(w)$ is a standard Young tableau.

Example

For example, $w = 452361$ *is good since*

$$SD(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

while $\pi = 13254$ *is not good, as*

$$SD(\pi) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Robinson-Schensted (RS) Correspondence

RS is a well-studied bijection

$$\pi \mapsto (\mathrm{P}(\pi), \mathrm{Q}(\pi))$$

between permutations and pairs of standard Young tableaux of the same shape.
$\mathrm{P}(\pi)$ is called the *insertion tableau* and $\mathrm{Q}(\pi)$ is called the *recording tableau*.

Example

$w = 452361.$

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

RS Example

Example

Let $w = 452361$. *Then let* P_k, Q_k *denote the insertion and recording tableaux, respectively, after* k *insertions. We have that*

$$P_1 = \boxed{4} \qquad Q_1 = \boxed{1}$$

$$P_2 = \boxed{4} \leftarrow 5 \qquad Q_1 = \boxed{1}$$

$$P_2 = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array} \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$P_3 = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array} \leftarrow 2 \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$P_3 = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline\end{array} \atop \leftarrow 4 \qquad Q_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 5 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

SD for FC permutations

Definition 7

A permutation w is *fully commutative (FC)* if its P(w) tableau has height at most 2 (at most 2 rows).

Example

$w = 14253$ *is FC as*

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

while $v = 52341$ *is not FC:*

$$P(v) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

SD for FC permutations

Definition 8

The *row reading word* of a tableau T, denoted $\text{rrw}(T)$, is the permutation produced by reading the entries of T from bottom to top, left to right.

Example

The row reading word of

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 7 & 8 \\ \hline 2 & 5 & 6 & 9 \\ \cline{1-4}\end{array}$$

is $\text{rrw}(P) = 256913478$.

SD for FC permutations

Proposition 9

For any FC permutation w, $\mathrm{rrw}(\mathrm{P}(w)) = \mathrm{rrw}(\mathrm{SD}(w))$.

Example

Consider $w = 215369478$.

$$P(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 7 & 8 \\ \hline 2 & 5 & 6 & 9 & \\ \cline{1-4}\end{array} \qquad SD(w) = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 7 & 8 \\ \hline 6 & 9 \\ \cline{1-2} 5 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$P(w) \neq SD(w)$ *but* $\mathrm{rrw}(P(w)) = 256913478 = \mathrm{rrw}(SD(w))$*!*

SD for FC permutations

Proposition 10

For tableau T, T is the SD of an FC permutation iff $\exists$ FC Tableau P such that $\text{rrw}(T) = \text{rrw}(P)$ and $row1(T) = row1(P)$.

Example

$T =$

1	3	4	7	8
6	9			
5				
2				

$P =$

1	3	4	7	8
2	5	6	9	

Construct w as follows: $2569 \rightarrow w = 21\ 53\ 6947\ 8$

$t = 0$	...		2	1	5	3	6	9	4	7	8								...
$t = 1$	...			2		5			6	9		1	3	4	7	8			...

Good tableaux

Recall the RS correspondence introduced earlier that there is a bijection between permutations and pairs standard tableaux, the second of which is called the recording tableau.

Theorem 11 (Cofie, Fugikawa, Gunawan, Stewart, Zeng 21)

If $Q(w) = Q(\pi)$, *then* w *is good if and only if* π *is good*

In view of this theorem,

Definition 12

We call a tableau T good if for any w with $Q(w) = T$, w is good.

Motzkin recursion and good tableaux

Definition 13

A sequence of finite sets of objects $A_0, A_1, A_2, \ldots$ indexed by n is said to be a *Motzkin object* if $|A_0| = |A_1| = 1$ and for $n \geq 2$,

$$|A_n| = |A_{n-1}| + \sum_{k=0}^{n-2} |A_k| \cdot |A_{n-k-2}|$$

Last year's group computed that, up to $n = 13$, the good tableaux of size n were counted by the Motzkin numbers. Our plan is as follows:

- Define a parallel operation on tableaux for every operation in the Motzkin recursion
- Prove that these operations are goodness preserving
- Prove that any portion of the recursion on tableaux generates a unique tableau in the family
- Prove that all good tableaux belong to the recursive family generated by these operations

A special case of tableaux multiplication

Definition 14

We define a non-commutative notion of multiplication on two standard tableaux that we call *tilde multiplication*, an example of which is illustrated below.

Example

Consider $T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$, $T_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. *Then we define* $\overline{T_1} = \begin{array}{|c|c|c|}\hline 3 & 4 & 7 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$. *Then we compute that*

$$T_1 \widetilde{\times} T_2 = \overline{T_1} \times T_2 = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Tilde multiplication preserves goodness

Theorem 15

Suppose T_1, T_2 are standard tableaux. Then, $T_1 \widetilde{\times} T_2$ is good if and only if T_1 and T_2 are good.

One of the primary lemmas to prove this theorem is below.

Lemma 16

If $\pi_1 = \mathrm{RS}^{-1}(T_1, T_1)$ and $\pi_2 = \mathrm{RS}^{-1}(T_2, T_2)$, then

$$Q(\overline{\pi_2} \cdot \pi_1) = T_1 \widetilde{\times} T_2$$

where $\overline{\pi_2}(k) = \pi_2(k) + n_1$ where n_1 is the size of T_1.

Column bump

Definition 17

Let T be a standard tableau. We define the *column bump* of T, denoted bump(T), to be $T \widetilde{\times} \boxed{1}$.

Example

$$\text{Take } T_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}, \quad T_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}.$$

$$\text{Then bump}(T_1) = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}, \quad \text{bump}(T_2) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$$

Row wrap

Definition 18

Let T be a standard tableau. We define the *row wrap* of T, denoted $\text{wrap}(T)$, by constructing a tableau by increasing every element of T by 1 to get T', and then replacing the first row of T' with a copy of the first row of T' but prepended by 1 and appended by $n+2$, where n is the size of T. Note that $\text{wrap}(T)$ is standard.

Example

$$\textit{Take } T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}, \quad T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}.$$

$$\textit{Then wrap}(T) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 8 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}, \quad \textit{wrap}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}$$

Bump and wrap preserve goodness

Theorem 19

Suppose Q is a standard tableau. Then, the following are equivalent:

(i) *Q is good*

(ii) *$bump(Q)$ is good*

(iii) *$wrap(Q)$ is good*

Recursive family of good tableaux

Definition 20

Let $K_0 = \{\varnothing\}$ and $K_1 = \left\{\boxed{1}\right\}$ where $\varnothing$ denotes the empty tableau. Then, we recursively define K_n as follows:

- for each $Q \in K_{n-1}$, $\text{bump}(Q) \in K_n$
- for each pair of $Q_1 \in K_k$ and $Q_2 \in K_{n-k-2}$ for $0 \leq k \leq n-2$, $Q_1 \tilde{\times} \text{wrap}(Q_2) \in K_n$

No other tableaux are in K_n.

Example

$$K_2 = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right\}, \quad K_3 = \left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right\}$$

By the above theorems, K_n consists strictly of good tableau of size n for $n \geq 1$.

K_n as a Motzkin object

Recall the Motzkin recursion.

Definition

A sequence of finite sets of objects $A_0, A_1, A_2, \ldots$ indexed by n is said to be a Motzkin object if $|A_0| = |A_1| = 1$ and for $n \geq 2$,

$$|A_n| = |A_{n-1}| + \sum_{k=0}^{n-2} |A_k| \cdot |A_{n-k-2}|$$

Theorem 21

The K_n tableaux are a Motzkin object.

A reverse inclusion

While generating a family of good tableaux which are a Motzkin object relies on existing machinery, the bulk of the below theorem lies in the reverse inclusion. We developed some new machinery to do so.

Theorem 22

A tableau T of size n is good if and only if $T \in K_n$. In particular, good tableaux of size n are a Motzkin object.

T	*H*	*A*	*N*	*K*
Y	*O*	*U*	!	

Any questions?