Box-Ball Systems and RSK Recording Tableaux

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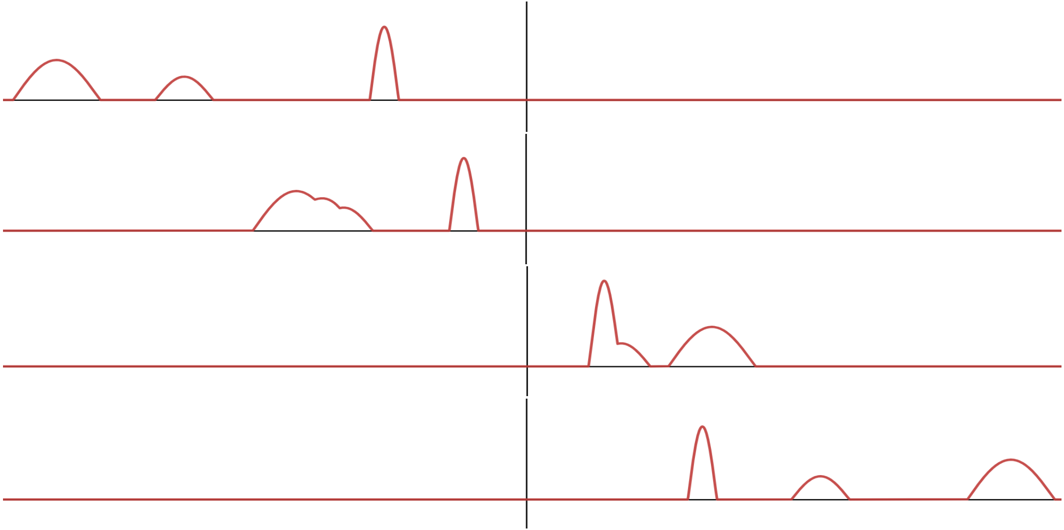
Motivation

(Desmos link)



Box-Ball System

Definition

The *box-ball system* (BBS) is a dynamical system consisting of a finite amount of numbered balls in an infinite strip. Balls take turns jumping to the first available cell, beginning with the smallest-numbered ball.

Example

One possible starting configuration of a box-ball system:

4	5	2	3	6	1								

We often denote empty boxes with e, and the state of a box-ball system as a string:

$$452361eeeeeeee$$

Box-Ball Move Example ($t = 1$)

At $t = 0$, consider the configuration 452361$eee\ldots$

$t = 0$	4	5	2	3	6	1			
	4	5	2	3	6		1		
	4	5		3	6	2	1		
	4	5			6	2	1	3	
		5	4		6	2	1	3	
			4	5	6	2	1	3	
$t = 1$			4	5		2	1	3	6

So at $t = 1$, the configuration is $ee45e2136$.

Box-Ball System Example ($t = 0$ through 5)

Starting with configuration 452361*eeee*, we have

$t = 0$	4	5	2	3	6	1															
$t = 1$			4	5		2	1	3	6												
$t = 2$					4	5	2			1	3	6									
$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

Solitons and Steady State

Definition

A *soliton* of a box-ball system is a consecutive maximal increasing subsequence of balls that do not interact with other balls in any future box-ball move.

Example

The strings 4, 25, and 136 are solitons, because they do not interact in future box-ball moves:

$$t = 3 : \ldots e425eee136eeeeeee \ldots$$
$$t = 4 : \ldots ee4e25eeee136eeee \ldots$$
$$t = 5 : \ldots eee4ee25eeeee136e \ldots$$

Solitons and Steady State

Definition

A box-ball system is in *steady state* if the system is decomposed into solitons, i.e., each non-empty ball belongs to one soliton. The number of moves it takes for a permutation to reach steady state is called its *steady-state time* (SST).

Example

452361 decomposed into solitons 4, 25, and 136 at $t = 3$, so its SST is 3.
1423567 decomposes into solitons at $t = 1$, so its SST is 1:

$$t = 1 : e4eee312567eeeee$$

$$t = 2 : ee4eee3eeee12567$$

Theorem

Every permutation reaches steady state eventually.

Questions

- How can we determine the time to steady state of a box-ball system?
- What does the steady state of a box-ball system look like? How do the balls break down into solitons?

Tableaux

Definition

A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing. A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

1	2	4
3	5	
6	7	

1	3	6
2	5	
4		

1	3	4
2		
5		
6		

Nontableau:

1	2	
3	5	4

Nonstandard Tableau:

1	3	2
6	4	7
5		

Soliton Decomposition

Definition

The *soliton decomposition*, denoted SD, of a permutation is a tableau of its solitons "stacked from right to left."

Example

The following are a few BBS configurations for 452361:

$$t = 3 : \ldots e425eee136eeeeeee\ldots$$

$$t = 4 : \ldots ee4e25eeee136eeee\ldots$$

$$t = 5 : \ldots eee4ee25eeeee136e\ldots$$

$$SD(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}$$

RSK Bijection

Theorem

The Robinson-Schensted-Knuth(RSK) algorithm is a bijection

$$\pi \mapsto (P(\pi), Q(\pi))$$

from S_n onto pairs of size-n standard tableaux of equal shape. The tableau $P(\pi)$ is called the insertion tableau of π. The tableau $Q(\pi)$ is called the recording tableau of π.

Example

Let $\omega = 25143$. $P(\omega) =$

1	3
2	4
5	

and $Q(\omega) =$

1	2
3	4
5	

.

Current Work

Result I

Let $\pi, \omega \in S_n$. If $Q(\pi) = Q(\omega)$, then

- π and ω have the same time to steady state, and
- the soliton decompositions of π and ω have the same shape

Example

$$\pi = 2143 \text{ and } \omega = 3142$$

$$Q(\pi) = Q(\omega) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

Both π and ω reach steady state at $t = 1$.

$$SD(\pi) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \qquad SD(\omega) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

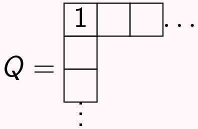
Current Work (I)

Open Question

Find a formula to send the recording tableau Q to the time to steady state.

Result II

All permutations that have Q of the shape



reach steady state at t=1.

Result III

All non-crossing involutions reach steady state after at most 1 time step.

Current Work (I)

Conjecture

For $n \geq 4$, the maximum time to steady state is $n-3$.

Proof Outline:

1. Rightmost soliton is formed after 1 box-ball move
2. This recording tableau maximizes time to steady state: $$Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \dots & n-2 & n-1 \\ \hline 3 & 4 \\ \cline{1-2} n \\ \cline{1-1} \end{array}.$$

Current Work (II)

Question

When is the soliton decomposition $SD(\omega)$ standard?

Theorem (UConn Math REU 2020)

Given $\omega \in S_n$, the following are equivalent:

- $SD(\omega)$ *is standard*
- $SD(\omega) = P(\omega)$
- *the shape of* $SD(\omega)$ *is the same as the shape of* $P(\omega)$

Current Work (II)

Recall

The recording tableau Q determines the shape of $SD(\omega)$.

Corollary

If $Q(\omega) = Q(\pi)$, then $SD(\omega) = P(\omega)$ if and only if $SD(\pi) = P(\pi)$.

Current Work (II)

Result IV

If $Q(\omega)$ satisfies $SD(\omega) = P(\omega)$, and if Q' is Q with its largest k cells removed, then Q' satisfies $SD(\pi) = P(\pi)$ as well.

Example

The recording tableau $Q =$

1	2	3	7
4	5	6	
8			

satisfies $SD(\omega) = P(\omega)$.

Then the "reduced" recording tableau $Q' =$

1	2	3
4	5	

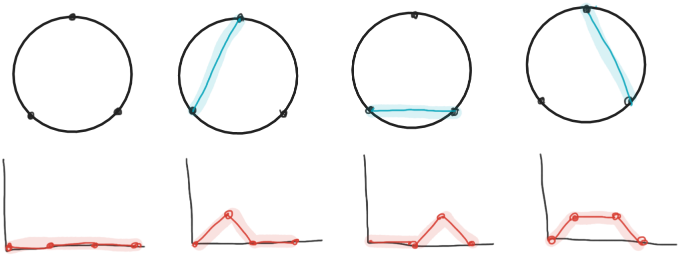
also satisfies $SD(\pi) = P(\pi)$.

Current Work (II)

Conjecture

$\{Q(\omega) : \omega \in S_n \text{ and } SD(\omega) = P(\omega)\}$ are counted by the Motzkin numbers.

Other objects counted by Motzkin numbers:



$n = 3$

Thank you!