First & Last Name: ______________________ Student ID: ______________

Linear Algebra Worksheet Section 4.2

1 T/F Questions

Let $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix}$.

(a) True or false? The set of all solutions to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$. ◯ True ◯ False

Briefly (in 1-2 sentences) justify why:

Solution: True, because this set is equal to Nul A (by definition of null spaces), which is a subspace of $\mathbb{R}^4$, since A has 4 columns (see Theorem 2 in Sec 4.2).

(b) True or false? The set of all solutions to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ is a subspace of $\mathbb{R}^4$. ◯ True ◯ False

Briefly justify why:

Solution: FALSE. This set does not contain the zero vector of $\mathbb{R}^4$, since $A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$.

Therefore this subset of $\mathbb{R}^4$ is not a subspace.

(c) Let H be the set of all vectors in $\mathbb{R}^4$ whose entries a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$.

True or false? H is a subspace of $\mathbb{R}^4$. ◯ True ◯ False

Briefly justify why:

Solution: True.

Why is this true? Observe that H is the set of all solutions of

$$\begin{Bmatrix} a - 2b + 5c - d = 0 \\ -a - b + c = 0 \end{Bmatrix}$$

which is a system of homogeneous linear equations in 4 unknowns. So H is a subspace of $\mathbb{R}^4$ (see Theorem 2 in Sec 4.2).

(d) Let H be the set of all vectors in $\mathbb{R}^4$ whose entries a, b, c, d satisfy the equations $a - 2b + 5c = d + 1$ and $c - a = b$.

True or false? H is a subspace of $\mathbb{R}^4$. ◯ True ◯ False

Briefly justify why:

Solution: True.

Why is this true? Observe that H is the set of all solutions of

$$\left\{\begin{matrix} a - 2b + 5c - d = 1 \\ -a - b + c = 0 \end{matrix}\right\}$$

which is a system of <u>non</u>-homogeneous linear equations. This subset of $\mathbb{R}^4$ is *not* a subspace because the zero vector of $\mathbb{R}^4$ is not in H.

Recall that $\mathbb{P}_2$ denotes the vector space of polynomials whose degree is at most 2.

Def. The kernel of a linear map T from V to W is the set of elements in the domain V that are sent by T to the zero object in the codomain W.

2 The kernel and range (image) of a linear transformation

Define a linear map $T : \mathbb{P}_2 \to \mathbb{R}^3$ by

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(5) \\ \mathbf{p}(5) \\ \mathbf{p}(5) \end{bmatrix}$$

For example, if $\mathbf{p}(t) = 3 + t^2$, then $T(\mathbf{p}) = \begin{bmatrix} 28 \\ 28 \\ 28 \end{bmatrix}$.

(a) What is the kernel of T? Find two polynomials in $\mathbb{P}_2$ that span the kernel of T.

Solution: The kernel of T is $\boxed{\text{the set of polynomials } p(t) \text{ in } \mathbb{P}_2 \text{ such that } p(5) = 0}$.

A possible spanning set of the kernel of T is $\boxed{\{t - 5,\ t^2 - 25\}}$.

(b) What is the range of T? Find a vector in $\mathbb{R}^3$ which spans the range of T.

Solution: Since all entries in an image $T(p)$ must be equal, an element in the range of T must be of the form $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$. Given any $r \in \mathbb{R}$, we see that the constant polynomial $p(t) = r$ is sent to $\begin{bmatrix} r \\ r \\ r \end{bmatrix}$. So the range of T is

$$\boxed{\left\{ \begin{bmatrix} r \\ r \\ r \end{bmatrix} : r \in \mathbb{R} \right\}}$$

A possible spanning set of the range of T is $\boxed{\left\{ \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} \right\}}$.

Solution: For a similar problem, see Sec 4.2 Exercise 44 (assigned in MML).