Sec 8.5 Polytopes

Applications: linear programming, game theory, optimization problems

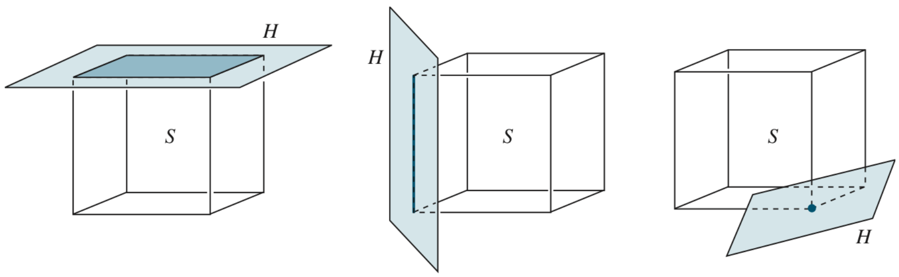
A **polytope** in $\mathbb{R}^n$ is the convex hull of a finite set of points. In $\mathbb{R}^2$, a polytope is simply a polygon. In $\mathbb{R}^3$, a polytope is called a polyhedron. Important features of a polyhedron are its faces, edges, and vertices. For example, the cube has 6 square faces, 12 edges, and 8 vertices.

EXAMPLE 1 Suppose S is a cube in $\mathbb{R}^3$. When a plane H is translated through $\mathbb{R}^3$ until it just touches (supports) the cube but does not cut through the interior of the cube, there are three possibilities for $\underbrace{H \cap S}_{\text{called "faces"}}$, depending on the orientation of H. (See Figure 1.)

$H \cap S$ may be a 2-dimensional square face (facet) of the cube.

$H \cap S$ may be a 1-dimensional edge of the cube.

$H \cap S$ may be a 0-dimensional vertex of the cube. ∎



$H \cap S$ is 2-dimensional. $H \cap S$ is 1-dimensional. $H \cap S$ is 0-dimensional.

Notation: If f is a linear functional, then $f(A) \leq d$ means $f(\mathbf{x}) \leq d$ for each $\mathbf{x} \in A$.

Def Let S be a polytope in $\mathbb{R}^n$ of dimension k.

- A nonempty subset F of S is called a (proper) face of S if $F \neq S$ and $F = S \cap H$ for some hyperplane $H = [f : d]$ such that either $f(S) \leq d$ or $f(S) \geq d$.

- The hyperplane H is called a supporting ("touching") hyperplane to S.
- If $\dim(F) = k$, then F is called a k-face of S

  → A 0-face of S is called a vertex of S,

  a 1-face of S is called an edge of S,

  and a $(k-1)$-face of S is called a facet of S

Theorem / Def

(i) A point $\vec{p}$ in P is a vertex of P iff

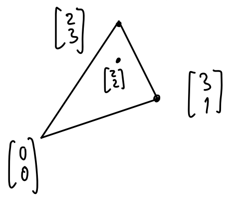
$\vec{p}$ is an <u>extreme point</u> of P

(<u>def</u> $\vec{p}$ is not in the interior of any line segment that lies in P,)

i.e. if $x, y \in P$ and $\vec{p} \in \overline{xy}$, then $x = \vec{p}$ or $y = \vec{p}$)

(ii) The set $\{v_1, \dots, v_k\}$ is the <u>minimal representation</u> of a polytope P if $P = \text{Conv}\{v_1, \dots, v_k\}$ and the vertices of P are $v_1, \dots, v_k$.

Ex: Let $P = \text{Conv}\left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix} \right\}$



The minimal representation of P is $\left\{ \begin{bmatrix}0\\0\end{bmatrix}, \cancel{\begin{bmatrix}2\\2\end{bmatrix}}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix} \right\}$

In summary ...

THEOREM 14

Suppose $M = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is the minimal representation of the polytope P. Then the following three statements are equivalent:

a. $\mathbf{p} \in M$.

b. $\mathbf{p}$ is a vertex of P.

c. $\mathbf{p}$ is an extreme point of P.

Thm (Thm 16)

Let f be a linear functional defined on a polytope S. Then there exist extreme points (i.e. vertices) $\hat{v}$, $\hat{\omega}$ of S such that

$f(\hat{v})$ = maximum value of f on S $\overset{\text{notation}}{=} \max_{v \in S} f(v)$ and $f(\hat{\omega}) = \min_{v \in S} f(v)$

Ex (Example 3(b) & (c)) In class we did (c)

Let $\vec{p}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\vec{p}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{p}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$, and $S = \operatorname{Conv}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$

Consider the linear functional $f : \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto -3x_1 + x_2$

(i) Find the maximum value m of f on S

(ii) Find all points $\vec{x}$ in S at which $f(\vec{x}) = m$.

Sol: By Thm 16, m is attained at one of the extreme points of S. (To find m, evaluate f at each extreme point and select the largest value)

After sketching, we see S is a triangle w/ vertices $\vec{p}_1, \vec{p}_2, \vec{p}_3$.

$$f(\vec{p}_1) = f\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = 3+0 = 3$$

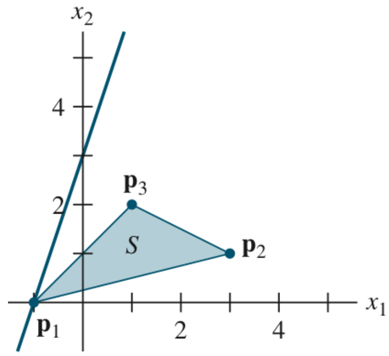
$$f(\vec{p}_2) = f\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = -9+1 = -8$$

$$f(\vec{p}_3) = f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = -3+2 = -1$$

(i) So $\boxed{m = 3}$.

Graph the hyperplane (line) $f(x_1, x_2) = m$ or $[f:m]$

$-3x_1 + x_2 = 3$



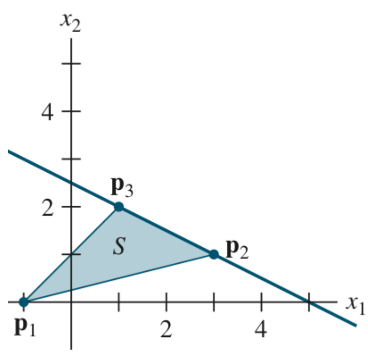
(b) $-3x_1 + x_2 = 3$

(ii) So $\vec{x} = \vec{p_1}$ is the only point in S at which $f(\vec{x}) = 3$.

Ex (iii) Do the same for linear functional $g(x_1, x_2) = x_1 + 2x_2$.

Sol:

$g(\mathbf{p}_1) = -1$, $g(\mathbf{p}_2) = 5$, and $g(\mathbf{p}_3) = 5$, so $m_3 = 5$. Graph the line $g(x_1, x_2) = m_3$, that is, $x_1 + 2x_2 = 5$. Here, g attains its maximum value at $\mathbf{p}_2$, at $\mathbf{p}_3$, and at every point in the convex hull of $\mathbf{p}_2$ and $\mathbf{p}_3$. See Figure 4(c). ■



(c) $x_1 + 2x_2 = 5$

In general, for a higher dimensional polytope S, the maximum value of a linear functional f on S occurs at the intersection of a supporting hyperplane and S. This intersection is itself a polytope S' whose vertices form a subset of vertices of S.

- A polytope S as the convex hull of a finite set of points is an "explicit" representation of S.
- S also has an "implicit" representation as the intersection of a finite number of (closed) half-spaces.

This next ex explains how to compute it.

EXAMPLE 4 Let

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \text{ let } S = \operatorname{conv}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$$

By inspection,

the line $L_1 = \operatorname{aff}\{\vec{p}_1, \vec{p}_2\}$ through $\vec{p}_1$ and $\vec{p}_2$ is $x_1 + x_2 = 1$,

and S is on the side of L_1 where $x_1 + x_2 \geq 1$,

since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is on the side where $x_1 + x_2 \leq 1$

the line $L_2 = \operatorname{aff}\{\vec{p}_2, \vec{p}_3\}$ through $\vec{p}_2$ and $\vec{p}_3$ is $x_1 - x_2 = 1$

and S is on the side of L_2 where $x_1 - x_2 \leq 1$

since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is on this same half-space

Find the line L_3 $\operatorname{aff}\{\vec{p}_1, \vec{p}_3\}$:

L_3 is parallel to the line L_0 through the origin whose normal vector $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to $\vec{p}_1 - \vec{p}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solve $\vec{n} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$: $3a + b = 0$, a possible answer $\vec{n} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

So L_0 is $-1x_1 + 3x_2 = 0$, $f(x_1, x_2) = -x_1 + 3x_2$

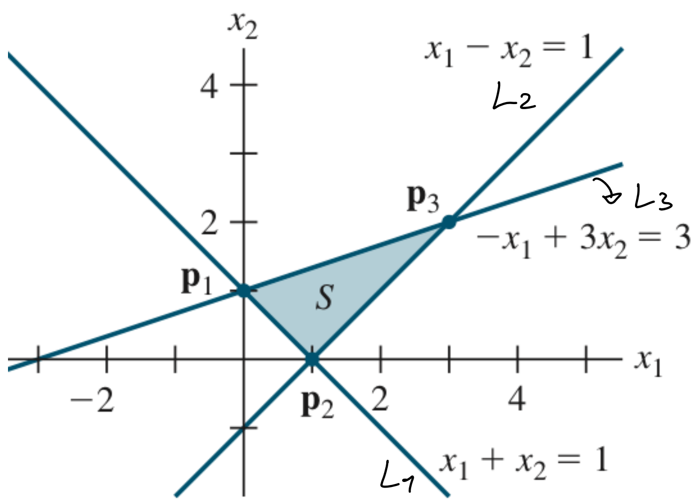
and L_3 is $-1x_1 + 3x_2 = d$ for some d

Since $\vec{p}_1$ is on line L_3, $d = f(\vec{p}_1) = f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3$

So L_3 is $-x_1 + 3x_2 = 3$

S is in the half-space where $-x_1 + 3x_2 \leq 3$

since $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is on the same half-space.



Thus S can be $\underbrace{\text{described}}$ as the solution set of the system of linear inequalities

this is an implicit representation of S

$$x_1 + x_2 \geq 1$$
$$x_1 - x_2 \leq 1$$
$$-x_1 + 3x_2 \leq 3$$

(I can also write $-x_1 - x_2 \leq -1$ if I want to write all "$\leq$" which I do)

Rewrite this system as $A\vec{x} \leq \vec{b}$ where

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

To go the other way, sketch the system of inequalities

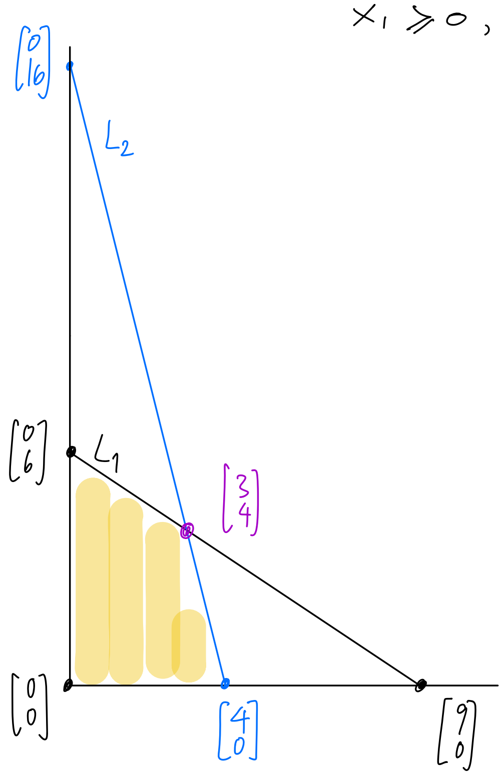
Exercise 6 : Find the minimal representation of

in MML the polytope defined by the inequalities

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 18 \\ 16 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqslant \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sol

The system is $2x_1 + 3x_2 \leq 18$ (L_1 is $2x_1+3x_2=18$)
$4x_1 + x_2 \leq 16$ (L_2 is $4x_1+x_2=16$)
$x_1 \geqslant 0, \quad x_2 \geqslant 0$



The vertices of S are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, so

the minimal representation of S is $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}$.

— the end —